1. Let the sequence \( \{a_n\} \) be given by \( a_1 = \sqrt{2} \) and \( a_{n+1} = \sqrt{2 + a_n} \).
   
   (a) Using the principle of mathematical induction (or otherwise), show that \( \{a_n\} \) is increasing and bounded above by 3.

   (b) Explain why \( \lim_{n \to \infty} a_n \) exists. Hint: look up the completeness axiom.

   (c) Find \( \lim_{n \to \infty} a_n \).

2. Let the sequence \( \{x_n\} \) be defined by \( x_1 = 5, x_2 = 11 \) and the recursive equation \( x_{n+1} - 5x_n + 6x_{n-1} = 0 \) (for \( n \geq 2 \)). Using the principle of mathematical induction show that \( x_n = 2^{n+1} + 3^n - 1 \) for \( n \geq 1 \).

3. Use the principle of mathematical induction to show that \( 5^{2n} - 6n + 8 \) is divisible by 9 for \( n \geq 1 \).

4. Use the principle of mathematical induction to show that
   \[
   \sum_{i=1}^{N} \frac{z_i}{z_j} \leq \sum_{i=1}^{N} |z_i|
   \]
   where \( z_i \in \mathbb{C} \) for each \( i \in \mathbb{N} \). What is the geometric interpretation of this result for \( N = 2 \)?

5. Find a possible formula for a sequence \( \{x_n\} \) which satisfies all the following conditions:
   
   - \( \{x_n\} \) is neither increasing or decreasing;
   - \( 0 < x_n < 1 \) for every \( n \in \mathbb{N} \);
   - \( x_n \neq 1/2 \) for every \( n \in \mathbb{N} \);
   - \( x_n \to 1/2 \) as \( n \to \infty \).

6. Let the sequence \( \{a_n\} \) be defined by \( a_n = \frac{4n + 3}{5n + 2} \).
   
   (a) Evaluate \( \lim_{n \to \infty} a_n \) by using the arithmetic of limits

   (b) Use the \( \varepsilon-N \) definition of the limit of a sequence to prove that the limit from the previous part is correct.

7. Use the arithmetic of limits to evaluate the limit of the following sequences or explain why the limit fails to exist.
   
   (a) \( \log(5n + 2) - \log(3n + 1) \)

   (b) \( \sqrt{n + 2} - \sqrt{n} \)

   (c) \( (-1)^n \)

   (d) \( \frac{\sin n}{n} \) (What theorem do you need to use?)

8. Find and sketch the domain of \( f(x, y) = \sqrt{4 - x^2 - y^2} \) and \( z = \sqrt{x^2 - y^2} \).

9. Sketch the fibres over \( \alpha = 0, \pm 1 \) for \( f(x, y) = x^2 + 4y \) and \( g(x, y) = xy \).

10. What are the fibres over \( \alpha \) of a real valued function \( f : \mathbb{R} \to \mathbb{R} \)? What is the graphical meaning for the fibre over 0?
11. Evaluate each limit shown or explain why it does not exist. Justify your claims.

(a) \( \lim_{(x,y) \to (1,1)} \frac{2xy}{x^2 + 2y^2} \)

(b) \( \lim_{(x,y) \to (0,0)} \frac{2xy}{x^2 + 2y^2} \)

12. Using the formal definition of limits, prove that

\[ \lim_{(x,y) \to (1,1)} f(x,y) = 7 \]

where \( f(x,y) = 4x + 3y \).

13. What does it mean to say that \( f(x,y) \) is continuous at \((a,b)\)?

14. For each of the following, find \( \frac{\partial z}{\partial x} \), \( \frac{\partial z}{\partial y} \), \( \frac{\partial^2 z}{\partial x^2} \), \( \frac{\partial^2 z}{\partial y^2} \), \( \frac{\partial^2 z}{\partial x \partial y} \), and \( \frac{\partial^2 z}{\partial y \partial x} \).

(a) \( z = x^3 + 7x^6y^2 + 8x \)

(b) \( z = x\sin y + y\cos x \)

(c) \( z = y\log x \)

15. The monthly payment, \( P \), on a loan is a function of three variables \( P = f(A,r,N) \) where \( A \) is the amount borrowed, \( r \) is the interest rate, and \( N \) is the number of years the loan was taken out for.

(a) \( f(92000,14,30) = 1090.08 \). What does this tell you, in financial terms?

(b) \( \left. \frac{\partial P}{\partial r} \right|_{(92000,14,30)} = 72.82 \). What is the financial significance of this statement?

(c) Would you expect \( \frac{\partial P}{\partial A} \) to be positive or negative? Why?

(d) Would you expect \( \frac{\partial P}{\partial N} \) to be positive or negative? Why?

16. A student was asked to find the equation of the tangent plane to the surface \( z = x^3 - y^2 \) at the point \((x,y) = (2,3)\). The student’s answer was \( z = 3x^2(x-2) - 2y(y-3) - 1 \).

(a) At a glance, how do you know this is wrong?

(b) What mistake was made?

(c) Answer the question correctly.

17. Find the directional derivative of \( z = x^2y \) at \((1,2)\) in the direction making an angle of \( 5\pi/4 \) with the \( x \) axis. In which direction is the directional derivative the largest?