1. Determine whether or not the following series converge.

   (a) \[ \sum_{n=1}^{\infty} \frac{n}{n^3 + 1} \]

   (b) \[ \sum_{n=1}^{\infty} \left( \frac{n}{3n+1} \right)^n \]

   (c) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \]

   (d) \[ \sum_{n=1}^{\infty} \frac{n^n}{(2n)!} \]

   (e) \[ \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{5^n n!} \]

2. You will show that the sequence

   \[ t_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \]

   has a limit, which is called the Euler constant, \( \gamma \). Incidentally, it is not known whether \( \gamma \) is rational or irrational.

   (a) Figure 1 shows a graph of \( y = f(x) = 1/x \). To the figure, add the left hand Riemann sums from \( x = 1 \) to \( x = n \). Interpret \( t_n \) as an area to show that \( t_n > 0 \) for all \( n \).

   (b) Interpret

   \[ t_n - t_{n+1} = (\log(n+1) - \log n) - \frac{1}{n+1} \]

   as a difference of areas to show that \( t_n - t_{n+1} > 0 \).

   (c) What can you conclude?

   ![Figure 1: Graph of \( y = f(x) = 1/x \) with left hand Riemann sums added](image)
3. You will show that
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2 \]

Let \( h_n \) and \( s_n \) be the partial sums of the harmonic and alternating harmonic series.

(a) Show that \( s_{2n} = h_{2n} - h_n \).

(b) From question 2 we have
\[ h_n - \log n \to \gamma \quad \text{as} \quad n \to \infty \]

and therefore
\[ h_{2n} - \log 2n \to \gamma \quad \text{as} \quad n \to \infty \]

Use these facts together with part (3a) to show that \( s_{2n} \to \log 2 \) as \( n \to \infty \).

4. Let the set \( F \) be all functions of the form \( f : \mathbb{R} \to \mathbb{R} \) with the usual meanings of addition of functions and multiplication by scalars. Which of the following sets are subspaces of \( F \)?

(a) The set of all functions \( y \) that satisfy the differential equation \( 2 \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + y = 0 \).

(b) The set of functions \( g \) which satisfy \( f \neq 0 \).

5. Let \( M_{n \times n} \) be the vector space of all \( n \times n \) matrices. Which of the following are subspaces?

(a) All symmetric \( n \times n \) matrices. If so, what is a set of matrices which span this subspace for \( n = 3 \)?

(b) All invertible \( n \times n \) matrices.

(c) All \( 2 \times 2 \) matrices \( A \) such that \( A^2 = A \).

6. Given that \( \{ \tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \} \) are linearly independent set of vectors in \( \mathbb{R}^3 \), explain whether or not
\[ \{ \tilde{v}_1 + \tilde{v}_2 + \tilde{v}_3, \tilde{v}_1 + 2\tilde{v}_2 + 3\tilde{v}_3, \tilde{v}_1 - \tilde{v}_2 - 2\tilde{v}_3 \} \]

form a linearly independent set.

7. Let \((a, b)\) and \((c, d)\) be two elements of \( \mathbb{R}^2 \). Show that if \( ad - bc = 0 \) then they are linearly dependent, and that if \( ad - bc \neq 0 \) then they are linearly independent.

8. Prove that the functions \( e^x \) and \( xe^x \) are linearly independent. Hint: consider the equation that they must satisfy if they are to be linearly dependent, differentiate it to obtain a second equation and ponder.

9. Are the vectors
\[ \tilde{v}_1 = (1, 1, 2, 4) \]
\[ \tilde{v}_2 = (2, -1, -5, 2) \]
\[ \tilde{v}_3 = (1, -1, -4, 0) \]
\[ \tilde{v}_4 = (2, 1, 1, 6) \]

linearly independent in \( \mathbb{R}^4 \)? Find a basis for the subspace \( V = \text{span}\{ \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4 \} \).

10. Find a basis for the subspace \( V \) of \( \mathbb{R}^5 \) defined by
\[ V = \{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = 3x_2 \text{ and } x_3 = 7x_4 \} \]