1. Given that \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \), find the Maclaurin series for \( \log(1+x) \). What is the interval of convergence for this series? Use the series you have obtained for \( \log(1+x) \) to calculate \( \log(1.1) \) to 3 decimal places? How many terms do you need to include?

2. (a) Evaluate the indefinite integral \( \int \sin(x^2)\,dx \) as an infinite series.

(b) Use the series to approximate the definite integral \( \int_0^1 \sin(x^2)\,dx \) to 3 decimal places.

3. Find the Taylor series for \( f(x) = \sin x \) at \( a = \pi/2 \). Prove that the series obtained represents \( \sin x \) for all \( x \).

4. The Bessel function of order 0, \( J_0 \), is defined by

\[
J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}
\]

(a) Find the domain of \( J_0 \).

(b) Show that \( J_0 \) satisfies the differential equation

\[
x^2 J''_0(x) + x J'_0(x) + x^2 J_0(x) = 0
\]

(c) Evaluate \( \int_0^1 J_0(x)\,dx \) correct to three decimal places.

(d) The Bessel function of order 1, \( J_1 \), is defined by

\[
J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n! (n+1)!}
\]

Show that \( J'_1(x) = -J_1(x) \).

These function first arose when Bessel solved Kepler’s equations for describing planetary motion. Since then these functions have been applied in many different physical applications, including the shape of a vibrating drumhead.
5. Show that the following set of vectors is a basis for $M_{2 \times 2}$:

\[
\begin{bmatrix} 3 & 6 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -12 & -4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Express \[
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\] as a linear combination of this basis.

6. Determine bases for the following subspaces of $\mathbb{R}^3$.

(a) The plane $3x - 2y + 5z = 0$.

(b) The line $x = 2t, y = -t, z = 4t$.

7. Show that the set $W$ of polynomials in $P_2$ such that $p(1) = 0$ is a subspace of $P_2$.

(a) Make a conjecture about the dimension of $W$. Can you explain yourself in any way?

(b) Confirm (or reject) your conjecture by finding a basis for $W$.

8. Indicate whether the statement is always true or sometimes false. Justify your answer with a logical argument or provide a counterexample.

(a) If $E$ is an elementary matrix, then $A$ and $EA$ must have the same nullspace.

(b) If $E$ is an elementary matrix, then $A$ and $EA$ must have the same row space.

(c) If $E$ is an elementary matrix, then $A$ and $EA$ must have the same column space.

(d) If $A\mathbf{x} = \mathbf{b}$ does not have any solutions, then $\mathbf{b}$ is not in the column space of $A$.

(e) The row space and nullspace of an invertible matrix are the same.

9. Prove that the row vectors of an $n \times n$ invertible matrix $A$ form a basis for $\mathbb{R}^n$.

10. Suppose that $A$ and $B$ are $n \times n$ matrices and $A$ is invertible. Invent and prove a theorem that describes how the row spaces of $AB$ and $B$ are related. Hint: what can you determine about $A$ in terms of elementary matrices?