1. Let $u(x, y) = \frac{1}{2}x^2 - \frac{1}{2}x^2y^2 + 4y^2$.

(a) If $x = t$ and $y = f(t)$ then find the total derivative of $u$ with respect to $t$. That is, find $\frac{du}{dt}$.

(b) If $x = s^2t$ and $y = t^2 + 3st$ then find $\frac{du}{ds}$ and $\frac{du}{dt}$.

2. **Theorem:** Suppose $f$ is a differentiable function of two variables. Then the maximum value of the directional derivative $D_u f(x)$ is $|f'(x)|$ and it occurs when $u$ has the same direction as the gradient vector $f'(x)$.

(a) Prove the theorem just stated. Hint: recall the definition of $D_u f$ as well as the definition of the dot product given in Tutorial 1.

(b) Hence find the direction that $g(x, y) = x^2e^y$ is changing the fastest at the point $(1, 0)$. What is the maximum rate of change?

(c) Find the equation of the tangent plane and the normal line to $g(x, y)$ from part (2b) at the point $(1, 0)$.

3. Let $f(x, y) = x^3 - 3xy^2$. Find all the critical points. Can you classify them? Explain. Show that the fibre of $f(x, y)$ over $\alpha = 0$ consists of three lines intersecting at the origin. Show that $f(x, y)$ alternates from positive to negative in the regions defined by the lines. Sketch a contour diagram for $f$ near $(0, 0)$. Hint: look at the fibres over $\alpha = \pm 1$ and $\pm 2$.

The graph of this function is called a *monkey saddle*.

4. Use the ideas of geometric series to express the number $3.1724724724\ldots$ as a ratio of integers.

5. Using the integral test, show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $0 < p \leq 1$.

6. Test the following series for convergence or divergence.

(a) $\sum_{n=1}^{\infty} n^2 2^{-n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n} \cos(n\pi)$

(c) $\sum_{n=1}^{\infty} \frac{3n^2 + n + 1}{n^5 + 1}$

(d) $\sum_{n=1}^{\infty} \frac{1}{(\log n)^3}$

7. Let $A$ be an $n \times n$ matrix with rows $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$. Prove that if $\det A \neq 0$ then $\{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n\}$ are linearly independent.

8. (a) Do you think it is possible to have a vector space with exactly two vectors in it? Explain.

(b) Do you think that it is possible for a vector space to have two different zero vectors? That is, is it possible to have two different vectors $\vec{0}_1$ and $\vec{0}_2$ such that they both satisfy the vector space axioms?
9. Express the following as linear combinations of \( p_1 = 2 + x + 4x^2, \) \( p_2 = 1 - x + 3x^2, \) and \( p_3 = 3 + 2x + 5x^2 \).

(a) \(-9 - 7x - 15x^2\) \hspace{1cm} (b) \(6 + 11x + 6x^2\) \hspace{1cm} (c) 0 \hspace{1cm} (d) \(7 + 8x + 9x^2\)

10. Let \( f = \cos^2x \) and \( g = \sin^2x \). Which of the following lie in the space spanned by \( f \) and \( g \)?

(a) \(\cos 2x\) \hspace{1cm} (b) \(3 + x^2\) \hspace{1cm} (c) 1 \hspace{1cm} (d) \(\sin x\) \hspace{1cm} (e) 0

11. Let \( P_2 \) be the vector space of polynomials of degree less than or equal to 2. Determine whether or not the following polynomials span \( P_2 \).

\[ p_1 = 1 - x + 2x^2, \quad p_2 = 3 + x, \quad p_3 = 5 - x + 4x^2, \quad p_4 = -2 - 2x + 2x^2 \]

12. Which of the following sets of vectors \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) in \( \mathbb{R}^n \) are subspaces of \( \mathbb{R}^n \)?

(a) All \( \mathbf{x} \) such that \( x_1 \geq 0 \).

(b) All \( \mathbf{x} \) such that \( x_1x_2 = 0 \).

(c) All \( \mathbf{x} \) such that \( x_2 \) is rational.

13. Let the set \( F \) be all functions of the form \( f : \mathbb{R} \to \mathbb{R} \) with the usual meanings of addition of functions and multiplication by scalars. Which of the following sets are subspaces of \( F \)?

(a) The set of functions \( g \) such that \( g(x^2) = g(x)^2 \).

(b) The set of functions \( g \) such that \( g(0) = g(1) \).

(c) The set of all polynomials of degree exactly equal to 2.

(d) The set of all functions of the form \( k_1 + k_2 \sin x \), where \( k_1 \) and \( k_2 \) are real numbers.

14. Show that the solution vectors of a consistent nonhomogeneous system of \( m \) linear equations in \( n \) unknowns do not form a subspace of \( \mathbb{R}^n \).

15. Recall that the lines through the origin are subspaces of \( \mathbb{R}^2 \). If \( W_1 \) is the line \( y = x \) and \( W_2 \) is the line \( y = -x \), is the union \( W_1 \cup W_2 \) a subspace of \( \mathbb{R}^2 \)? Explain.