Indecomposability: polyominoes and polyomino tilings

Simone Rinaldi and D. G. Rogers

Dedicated to the memory of Alberto Del Lungo (1965–2003)
— ever an enthusiast for polyominoes

1 Polyomino problems

When we were preparing our article [1] on the enumeration of stack polyominoes for publication in *The Mathematical Gazette*, we thought to look back to see what else had appeared in the *Gazette* on the subject of *polyominoes*. A *polyomino* is a finite collection of cells in the square grid with connected interior — so it is insufficient that cells be connected only corner to corner. Some authors require polyominoes to have *simply* connected interior, that is, to be without holes, as was appropriate, for example, in [1] for stack polyominoes. The classic text on polyominoes is Solomon Golomb's engaging book [2], first published in 1965, now supplemented at an even more accessible level by [3].

There were a few contributions, as mentioned in our list of references in [1], to which [4] might be added. But we were somewhat surprised to find fewer than might have been expected for a topic that would seem to lend itself to the classroom, although it appeared that it had found greater favour over in the companion publication, *Mathematics in Schools*. Perhaps algebra is such a powerful, ubiquitous, neutral language that the extra effort needed to explain a combinatorial context, or, indeed, to envisage a geometrical one, is a deterrent off-setting any supposed gain in illumination or stimulus.

John Silvester had been one of the previous authors to consider polyominoes [5]. As we found, he too had been stimulated by reading [6] to look further at recurrence relations [7], but along the lines of generalisations to other divisions and powers of the terms satisfying second order linear recurrence relations. So, his response [8] to [1] is worth pondering.

> *As a dyed-in-the-wool algebraist, I find the style of Rajesh and Leversha in [6], with 20 equations and few words per page, easier to penetrate than your style, with only one or two equations, but a great many words per page. But I suspect that the difference of style is inevitable with the type of calculation*

you are explaining. It never ceases to amaze me how combinatorialists can always find just the right thing to count. I have no idea how you start to find a set of objects to fit a given sequence. Do you just explore lots and lots of sets of objects and throw away the ones that do not lead anywhere?

Of course, it had been a pure coincidence that our investigations of stack polyominoes brought us to the odd terms in the Fibonacci sequences just when we caught [6].

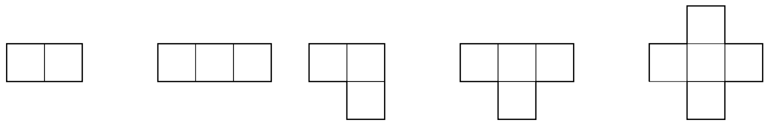


Figure 1: The five 2-indecomposable polyominoes.

However, we did also find two other items [9, 10] related to polyominoes that repaid further study, as we should like to report here. Among Nick MacKinnon's thoughts [10] on polyominoes in 1990 was the natural question of their *decomposability* into smaller polyominoes. Here, in common with MacKinnon, we have in mind to distinguish between polyominoes only up to reflections, rotations and translations. Clearly, all polyominoes can be dissected into single cells, as many as there are cells in the polyomino. But MacKinnon observed that, if the dissection is to be into polyominoes with at least two cells, then copies of the five polyominoes shown in Figure 1 are always sufficient for any polyomino itself with at least two cells, while these five polyominoes cannot further be decomposed. MacKinnon's proof was by induction. He then asked whether, on disallowing dominoes in these dissections, there is still a finite list of such indecomposable polyominoes sufficient for the dissection of any polyomino with at least three cells. His assertion that there is a list, in this case of 42 polyominoes, immediately arrested our attention. Having just been looking at the odd terms in the Fibonacci sequence, we naturally wondered whether, with this sequence of MacKinnon beginning

$$1, 5, 42 \ldots$$

we were now in the presence of the odd terms of the sequence of *Catalan numbers* C_n given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 0. \tag{1}$$

For the Catalan numbers are another old favourite, second only among integer sequences perhaps to the Fibonacci numbers in their popularity — Richard Stanley, for example, gives a whole menagerie of combinatorial interpretations in [11, Vol. II, Chap. 6, esp. q. 29, pp. 219–229; pp. 256–265] (see also sequence [12, *A*000108]). Here we were surprised to find that, while the Fibonacci sequence has appeared in the pages of the *Gazette* too many times to list here, the Catalan numbers had, in contrast, featured specifically only in two *Notes* [13, 14]. But they do make cameo appearances in [15, p. 106] and again in [16, p. 310], while a still-enigmatic generalisation had earlier been the subject of the fifth problem in [17, pp. 8–9, 286, 290]. There may now be some quickening of interest, since the current Editor of the *Gazette* gave a talk on the Catalan numbers at the annual conference of The Mathematical Association in 2003, while the appearance of Martin Griffiths' article [18] two issues ago more than makes up for any sense of past neglect in the *Gazette* itself (readers wanting to follow up on this type of combinatorial topic might look out for the same author's book on the central binomial coefficients published by UKMT this year).

Moreover, the generalisation of MacKinnon's problem was not only unrecognised in *The Online Encyclopedia of Integer Sequences* [12], but was also unknown to its progenitor and curator, Neil Sloane, and, for that matter, to Richard Stanley, too. So, here was something needing further investigation; and we make our report in the next section.

But, going back a further decade, we were also interested to see that Neil Rymer [9], in 1979, in describing the problem of enumerating *domino tilings* on the $4 \times n$ grid, alluded to a solution in terms of *indecomposable tilings*. Problems of this type retain interest, this particular instance resurfacing very recently in *The American Mathematical Monthly* [19]. Now, at about the same time as [9] appeared, it was pointed out more generally in [20] how they lead to *renewal* recurrence relations of the form

$$u_n = \sum_{r=1}^{n} f_r u_{n-r}, \quad n \geq 1; \quad u_0 = 1, \tag{2}$$

with, crucially, f_n positive. But such recurrence relations occur much more widely. For example, the special case

$$u_n = 2u_{n-1} + \sum_{r=2}^{n} u_{n-r}, \quad n \geq 2; \quad u_0 = 1,$$

appeared in *Student Problem 2003.3* [21] with $u_1 = 1$ and was given a combinatorial interpretation in terms of stack polyominoes in [1] when $u_1 = 2$. Moreover, the Catalan numbers themselves satisfy just such a recurrence relation:

$$C_n = \sum_{r=1}^{n} C_{r-1} C_{n-r}, \quad n \geq 1; \quad C_0 = 1. \tag{3}$$

Indeed, it is this *self-convolutional* nature of (3) that throws up the Catalan numbers in such a vast assortment of circumstances, just as the additive property governing the Fibonacci sequence occurs in many settings (see [12, *A*000048] and Section 4; and compare [18]). It seems worth elaborating on this in the *Gazette*, and we turn to this in Sections 3 and 4, looking respectively at the enumeration of lattice paths (complementing the approach in [18] in terms of bracketings) and of domino tilings.

2 Indecomposable polyominoes

For $n \geq 1$, let us say that a polyomino is *n-decomposable* if it can be dissected into smaller polyominoes each having at least n cells. A natural objective here is to refine these dissections until they can be refined no further. A polyomino with at least n cells that does not admit such a dissection is said to be *n-indecomposable*. Thus, any polyomino with at least n cells is either n-indecomposable or can be dissected into n-indecomposable polyominoes. For example, the first dissection shown in Figure 2 is into 3-indecomposable polyominoes; such dissections need not be unique, and, in this case, there is a dissection into a greater number of polyominoes all of which are 3-indecomposable. Notice also that these terms could equally well be defined relative to restricted classes of polyominoes, for instance polyominoes without holes or stack polyominoes.

In order to get a grip on MacKinnon's sequence, we need to show that, for any given $n \geq 1$, there is some threshold $h(n)$, such that any polyomino with at least $h(n)$ cells is n-decomposable. For, then there will only be a finite number, a_n, say, of polyominoes that are n-indecomposable, to be found as a subset of the finitely many polyominoes having less than $h(n)$ cells. Since the number p_n of polyominoes with n cells is known only for comparatively small n, this will not be much help in *determining* a_n. But it does confirm in general that MacKinnon will always produce a finite list of indecomposable polyominoes.

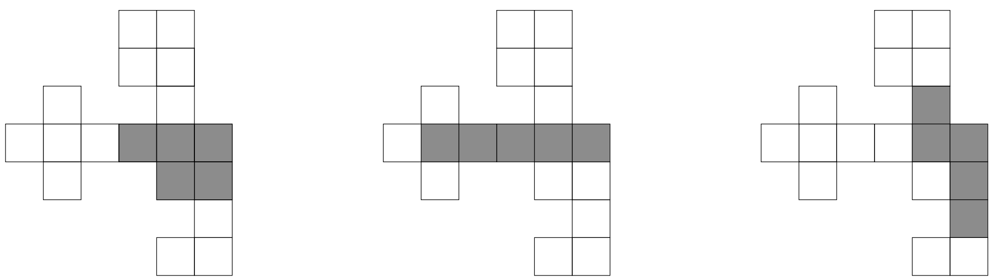


Figure 2: Connected components on removal of shaded cells.

Now, it does not matter for this purpose how crude a threshold we establish. Clearly a polyomino with at most $2n-1$ cells has too few cells to be n-decomposable. So, let us consider a polyomino with at least $2n$ cells. If we remove from this polyomino a connected subset of n cells including, for specificity, the top right-hand corner cell, that is, the cell highest up at the extreme right, then the polyomino will break up into connected components. As illustrated in Figure 2, there may be several ways to do this. Were any one of these connected components to have at least n cells, we could decompose the polyomino into that connected component and the rest of the polyomino, where both pieces might be further decomposable.

However, since we have only removed n cells in a connected group to begin with, their removal cannot produce too many connected components. Since each of the removed cells is connected to at least one other removed cell, it can contribute at most three edges to the boundary. So, the boundary of the removed portion has at most $3n$ edges, which means that there cannot be more than $3n$ connected components. Clearly, if the remnant of the polyomino contains at least $3n(n-1)+1$ cells, then one of the connected components has to contain at least n cells. It is therefore enough to take

$$h(n) \geq n + 3n(n-1) + 1, \quad n \geq 1,$$

to be assured that any polyomino with at least $h(n)$ cells is indeed n-decomposable. Hence, MacKinnon's sequence a_n is well-defined. Following [10], $a_1 = 1$ and $a_2 = 5$.

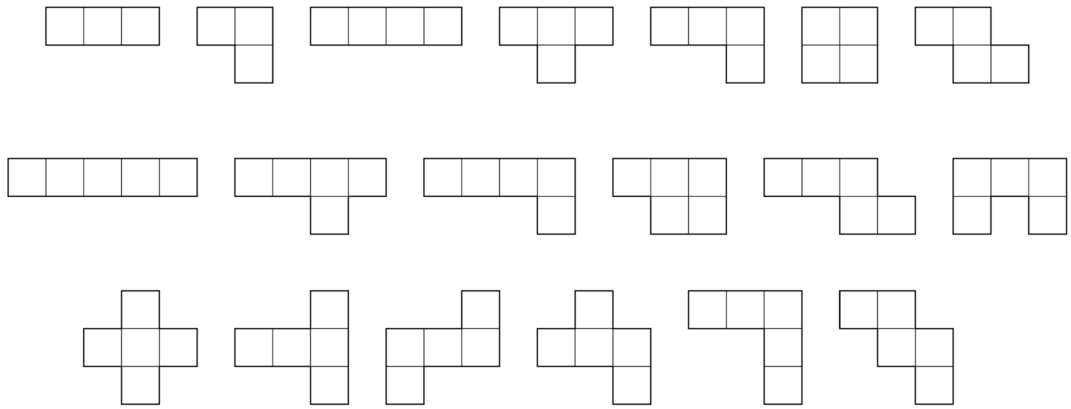


Figure 3: The 19 polyominoes with 3, 4 and 5 cells.

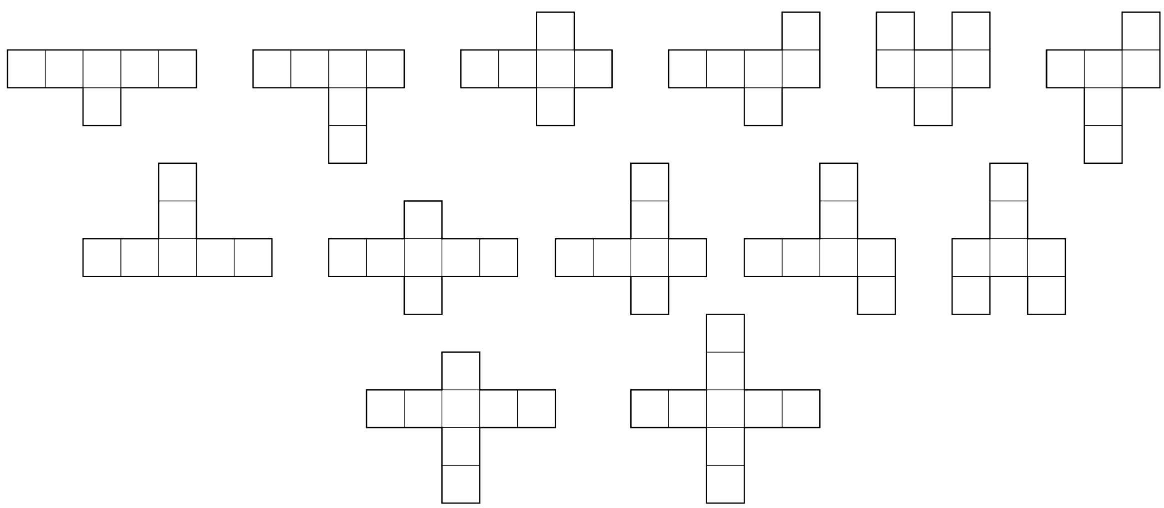


Figure 4: The 3-indecomposable polyominoes with 6, 7, 8 and 9 cells.

By adding one cell at a time in all possible positions on the boundary of any n-indecomposable polyomino, we can make an exhaustive search for n-indecomposable polyominoes for any specified n. This is still possible by hand for $n = 3$ without too much effort. Although $h(3) = 22$, in fact it turns out that all polyominoes with at least 10 cells are 3-decomposable.

No polyomino with fewer than 6 cells is decomposable in this way. As shown in Figure 3, there are 19 polyominoes in this category. Now, of the 35 polyominoes with 6 cells, only the 6 displayed in the top row of Figure 4 are 3-indecomposable. A polyomino with 6 cells has at most 14 boundary sites at which a further cell could be added to form a polyomino with 7 cells. So, it is only necessary to check at most $6 \times 14 = 84$ possibilities, instead of the full list of 108 polyominoes with 7 cells, to find the 5 that are 3-indecomposable (see the middle row of Figure 4). After this, our search introducing one new cell at a time quickly terminates, identifying in succession

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$a_1(m)$	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$a_2(m)$	0	1	2	1	1	0	0	0	0	0	0	0	0	0
$a_3(m)$	0	0	2	5	12	6	5	1	1	0	0	0	0	0
$a_4(m)$	0	0	0	5	12	35	108	73	76	80	25	15	15	0

Table 1: The numbers $a_n(m)$ of n-indecomposable polyominoes with m cells for $n = 1, 2, 3$ and 4.

unique 3-indecomposable polyominoes with 8 and 9 cells, but none thereafter (see the bottom row of Figure 4). There are at most 5×16 possibilities to check, then 1×18 and finally 1×20. This is perhaps just as well, since, in contrast, the numbers of polyominoes with 8, 9 and 10 cells are 369, 1285 and 4655 respectively, making for much longer searches. The third row of Table 1 summarises the tally of 3-indecomposable polyominoes with m cells. The total number of 3-indecomposable polyominoes is thus the row sum $a_3 = 32$. Our findings are therefore in *disagreement* with [10], dashing any conjecture that we might more generally be in the presence of C_{2n+1} — we had been uncertain whether MacKinnon had intended his remark, with his jaunty allusion to "*The Hitch-hikers Guide to the Galaxy*", as something of a mathematical joke, but he has since confirmed that it was just a mistake [22].

Looking back now at Figure 4, a more structural approach comes to mind. For, as we see, and can then argue, all 3-indecomposable polyominoes with at least 6 cells are *star-like*, in the sense that they have no cycles or 2×2 blocks of cells and there is exactly one cell adjacent to more than 2 other cells. Naturally a result like this is contingent on size. But, in similar spirit, a 4-indecomposable polyomino with at least 8 cells can be shown either to have a 2×2 block of cells or to be *tree-like* in having no cycles or such blocks of cells; there may be up to 3 cells adjacent to more than 2 other cells. Armed with these observations, straightforward case-by-case enumeration yields $a_4 = 444$ (see [12, *A*125709] for further terms in MacKinnon's sequence obtained by exhaustive computer search). Indeed, pushing this line of argument further reveals that, if an n-indecomposable polyomino has at least $3n-1$ cells, then there is a cell, deletion of which produces connected components each with at most n cells. Since there can be at most four such components, it is then immediate that a polyomino is n-decomposable if and only if it has at least $4n-2$ cells — a marked improvement on our earlier rough and ready threshold of $3n^2 - 2n + 1$ cells used to confirm that MacKinnon's sequence is well-defined.

3 Many happy returns

It is familiar that, if we consider the paths on the grid lines of the square grid, starting from the origin and moving always up or to the right, then we obtain an instance of Pascal's triangle, as in Figure 5(i). In particular, the number of such lattice paths to the point (m, n) is the binomial coefficient

$$\binom{m+n}{m}.$$

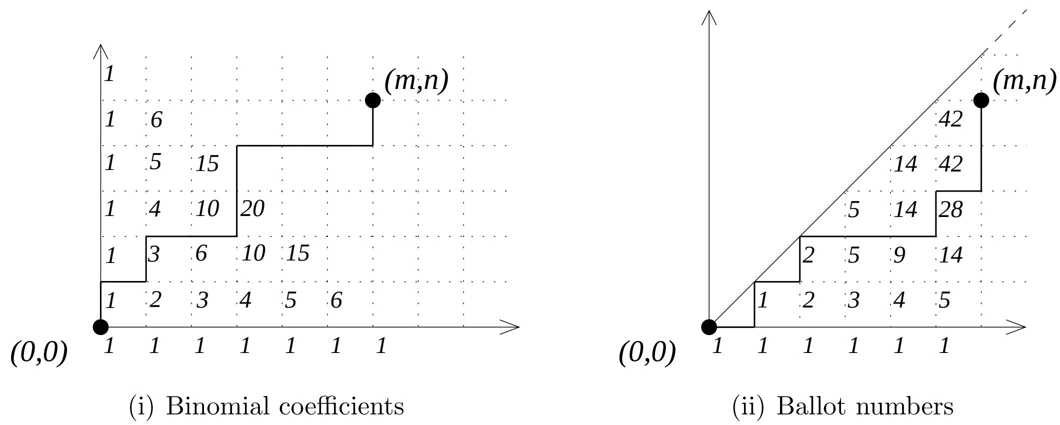


Figure 5: Counting lattice paths on the square grid.

However, imposing the condition that paths must not go above the main diagonal gives the array of *ballot numbers* $b(m, n)$, so called because they give the number of ways of tallying m votes for candidate A and n votes for candidate B where, throughout the counting, candidate A always has at least as many votes as candidate B (compare [23, 15, 16, 18] for some earlier appearances of this problem in the *Gazette*; and see Figure 5(ii)). In common with Pascal's triangle, there is the underlying recurrence relation

$$b(m, n) = b(m, n-1) + b(m-1, n), \tag{4}$$

but now this holds only for $0 < n < m$, subject to the boundary condition

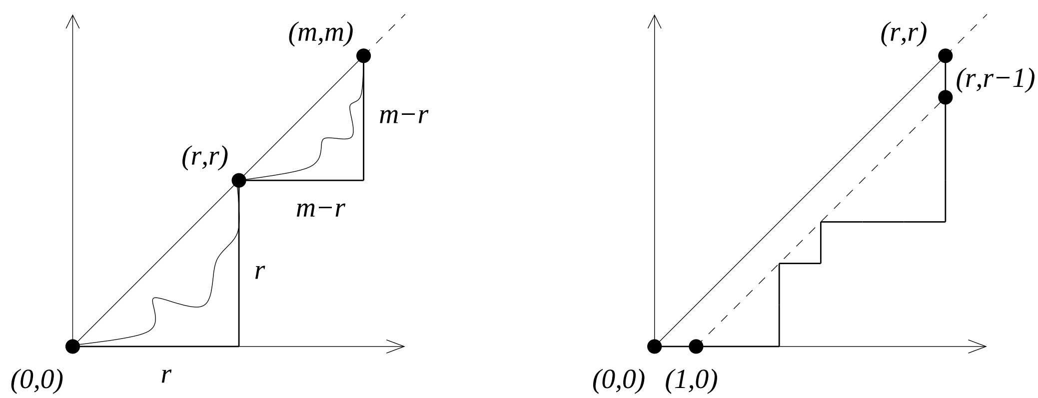
$$b(m, m) = b(m, m-1), \quad m > 0; \quad b(m, 0) = 1, \quad m \geq 0. \tag{5}$$

Indeed, (4) and (5) recapture in terms of these lattice paths the recurrence relation and boundary conditions explored in [13, 14] under other combinatorial interpretations. For that matter, recurrence relations similar to (4) featured much earlier in [24] explicitly in regard to lattice path enumeration, while material in other articles, such as [25, Table 3], is amenable to reinterpretation in these terms; on the other hand, [23] manages to avoid recurrence relations by working immediately with sums of binomial coefficients.

Looking at Figure 5(ii), we also recognize the Catalan numbers on the main diagonal. As a generalisation of (1), we have

$$b(m, n) = \frac{m-n+1}{m+1}\binom{m+n}{m}, \quad 0 \leq n \leq m. \tag{6}$$

Of course, (4) lends itself to an inductive proof of (6), based on (5), rather as in [13, 14]. Another popular approach is the "*reflection principle*" favoured in [16, 18] and commonly linked, as in [11, Vol. II, p. 212] and [26, 2nd. ed., p. 70], with the name of Désiré André (1840–1917) — Martin Griffiths [18] falls in-line with this appellation whereas, interestingly enough, Ray Hill [16] had earlier expressed reservations as to the attribition. Such doubts have been clarified only very recently in [27]. As reported there, André proceeds differently, without mention of this geometrical formulation, and there is no direct connection between what André does and the reflection principle, even if they lead to the same enumerative result. For that matter, it does not seem clear when the latter first appeared in the context of lattice path enumeration, still less how it became identified with André. Several other methods are available, including fairly direct combinatorial reasoning, bringing out the fraction in (6) (see [28] for a very recent review of proof techniques and [11] for further details; some historical account of the *generalised* Catalan numbers appearing in [18] is also presented in [27, 28]). Since the binomial coefficient in (6) counts the number of unrestricted paths, this fraction has a striking probabilistic meaning: if m votes for candidate A and n votes for candidate B are tallied at random, the probability that A stays abreast of B throughout the count is $(m-n+1)/(m+1)$.



(i) (ii)

Figure 6: Decomposition according to first return to main diagonal.

Our purpose here is not so much the derivation of (6), but rather to demonstrate the renewal recurrence relation (3) in terms of restricted lattice paths. For this, we borrow an observation that proves useful in probability theory, especially in connection with Markov chains: if an event occurs, there is a *first* time at which it occurs (compare, for example, [26]). So, if a restricted lattice path ends on the main diagonal at (m, m), there must be some least r, with $1 \leq r \leq m$, such that the path reaches the main diagonal at (r, r), as suggested in Figure 6(i). This allows us to classify the $b(m, m)$ walks in Figure 6(i) ending at (m, m) according to this *first return* to the main diagonal. If $\hat{b}(r, r)$ is the number of restricted paths to (r, r) that visit the main diagonal only at their beginning and end, then the number of restricted paths to (m, m) that make their first return to the main diagonal at (r, r) is

$$\hat{b}(r, r)b(m-r, m-r), \quad 1 \leq r \leq m,$$

since, having reached the main diagonal at (r, r), there are $b(m-r, m-r)$ ways to continue on to (m, m) by the translational invariance of the grid. Hence, considering all possible first returns to the main diagonal,

$$b(m,m) = \sum_{r=1}^{m} \hat{b}(r,r) b(m-r, m-r), m \geq 1.$$

But the translational invariance of the grid also ensures that (see Figure 6(ii))

$$\hat{b}(r,r) = b(r-1, r-1), \quad r \geq 1.$$

It follows that $b(m, m)$ satisfies the convolution (3), thus allowing the identification

$$b(m,m) = C_m, \quad m \geq 0.$$

Readers might like to try their hand at this style of argument by deducing that

$$b(m,n) = \sum_{r=0}^{n} b(r,r) b(m-r-1, n-r), \quad 0 \leq n \leq m, \tag{7}$$

from a classification of restricted paths according to their *last* visit to the main diagonal. Notice that, on introducing the generating functions

$$C(x) = \sum_{n \geq 0} C_n x^n; \quad B_k(x) = \sum_{n \geq 0} b(n+k-1, n) x^n, \quad k \geq 1,$$

so that $B_1(x) = C(x)$, then (3) and (7) take a simpler, attractive form. For, (3) becomes the functional equation

$$C(x) = 1 + xC(x)^2,$$

while (7) becomes

$$B_k(x) = B_1(x) B_{k-1}(x) = C(x) B_{k-1}(x), \quad k \geq 2,$$

revealing the ballot numbers to be the coefficients of the powers of $C(x)$. Indeed, this is one route to (1) and (6) by means of the analytical technique known as *Lagrange inversion* which applies to just this type of functional equation. However, the merit of these observations on generating functions lie more in their application to related enumeration problems, reducing them to computations in powers of $C(x)$.

4 Indecomposable domino tilings

What is the analogue for domino tilings of visits to the main diagonal that, as seen in the previous section, allow restricted lattice paths to be decomposed so effectively for purposes of enumeration? Clearly domino tilings fall apart in just this way in the presence of *faults* in the sense of [29], that is, grid lines not straddled by any dominoes. We can scan a domino tiling of the $k \times n$ square grid for these breaks much as we might scan a bar code. In particular, we first identify the least r, with $1 \le r \le n$, such that the given domino tiling breaks up into tilings on the $k \times r$ subgrid at the left and on whatever remains to the right of this, subjecting this latter portion to the same scrutiny for faults. If $r = n$, that is, there are no such breaks, then the domino tiling is said to be *indecomposable*. Thus, if $f_k(n)$ is the number of domino tilings of the $k \times n$ grid and $\hat{f}_k(n)$ of these tilings are indecomposable, then we have the renewal recurrence relation

$$f_k(n) = \sum_{r=1}^{n} \hat{f}_k(r) f_k(n-r), \quad n \ge 1; \quad f_k(0) = 1, \tag{8}$$

obtained by classifying the domino tilings according to their left-most fault.

Figure 7: $\hat{f}_2(1) = \hat{f}_2(2) = 1$.

The case $k = 2$ is especially familiar, since it gives one of the numerous instances of the Fibonacci numbers. From Figure 7, it is clear that the only indecomposable domino tilings of the $2 \times n$ grid occur for $n = 1, 2$, and

$$\hat{f}_2(1) = \hat{f}_2(2) = 1; \quad \hat{f}_2(n) = 0, \quad n \ge 2.$$

Hence, in this case, (8) becomes the usual recurrence relation for the Fibonacci numbers,

$$f_2(n) = f_2(n-1) + f_2(n-2), \quad n \ge 3,$$

with $f_2(1) = 1$ and $f_2(2) = 2$.

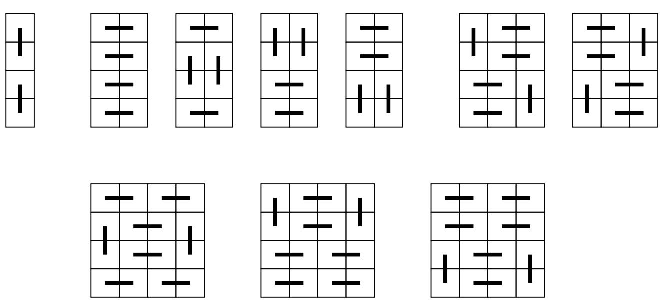


Figure 8: Indecomposable domino tilings of the $4 \times n$ grid for $1 \le n \le 4$: $\hat{f}_4(1) = 1$, $\hat{f}_4(2) = 4$, $\hat{f}_4(3) = 2$, $\hat{f}_4(4) = 3$.

But, for Rymer's case in [9], where $k = 4$, there are indecomposable domino tilings of the $4 \times n$ grid for all $n \ge 1$, so that (8) is a genuine renewal recurrence relation of the sort appearing in *Student Problem 2003.3* [21], as well as in [1]. Indeed, by inspecting Figure 8, we see that

$$\hat{f}_4(1) = 1; \quad \hat{f}_4(2) = 4; \quad \hat{f}_4(2n-1) = 2, \quad \hat{f}_4(2n) = 3, \quad n \ge 2,$$

so that (8) becomes

$$f_4(n) = f_4(n-1) + 4f_4(n-2) + 2f_4(n-3) + 3f_4(n-4) + 2f_4(n-5) + 3f_4(n-6) + \cdots ,$$

for n sufficiently large. Nevertheless, the pattern of coefficients here is sufficiently simple for the trick of differencing to produce an ordinary recurrence relation with constant coefficients. On writing $f(n) = f_4(n)$ for temporary ease of notation, we have:

$$\begin{array}{rl} f(n) & = f(n-1) + 4f(n-2) + 2f(n-3) + 3f(n-4) + 2f(n-5) + \cdots \\ f(n-2) = & \qquad\qquad\qquad\qquad\quad f(n-3) + 4f(n-4) + 2f(n-5) + \cdots \\ \hline f(n) - f(n-2) & = f(n-1) + 4f(n-2) + \ \ f(n-3) - \ \ f(n-4) \end{array}$$

Thus,

$$f_4(n) = f_4(n-1) + 5f_4(n-2) + f_4(n-3) - f_4(n-4), \quad n \geq 4,$$

with

$$f_4(0) = f_4(1) = 1, \quad f_4(2) = 5, \quad f_4(3) = 11,$$

in agreement with [9, 20].

For $k \geq 5$, the coefficients $\hat{f}_k(r)$ in (8) lack the simple regularity seen in the case $k = 4$. However, it can still be shown that $f_k(n)$ satisfies an ordinary linear recurrence relation with constant coefficients, by means of an enumerative analogue of Markov chains known as the *transfer matrix method* (see [11, Vol. I, Chap. 4, esp. q. 36, pp. 273–274, 291–292]). This approach can also be applied to many other polyomino tiling problems.

References

[1] S. Rinaldi and D. G. Rogers, How the odd terms in the Fibonacci sequence stack up, *Math. Gaz.*, **90**, (2006), 431–442.

[2] S. W. Golomb, *Polyominoes: Puzzles, Patterns, Problems, and Packings* (Scribner's, New York, NY, 1965; 2nd. ed., Princeton University Press, Princeton, NJ, 1994).

[3] G. E. Martin, *Polyominoes: A Guide to Puzzles and Problems in Tiling* (Math. Assoc. Amer., Washington, DC, 1991).

[4] R. B. J. T. Allenby, Note 83.54: Almost pavings of patios, *Math. Gaz.*, **83** (1999), 490–492.

[5] J. R. Silvester, Note 86.02: Painting by numbers or polyominoes revisited, *Math. Gaz.*, **86** (2002), 68–72.

[6] V. Ragesh and G. Leversha, Note 88.02: Some properties of the odd terms of the Fibonacci sequence, *Math. Gaz.*, **88** (2004), 85–86.

[7] J. R. Silvester, Note 90.33: The r-subsequences of the Fibonacci sequence, *Math. Gaz.*, **90** (2006), 263–266.

[8] J. R. Silvester, personal communication, July, 2004.

[9] N. W. Rymer, Projects, problems and patience, *Math. Gaz.*, **63** (1979), 1–7.

[10] N. I. P. MacKinnon, Some thoughts on polyomino tiles, *Math. Gaz.*, **74** (1990), 31–33.

[11] R. P. Stanley, Enumerative Combinatorics (Vol. I, Wadsworth, Belmont, CA, 1986; Vol. II, Cambridge University Press, Cambridge, UK, 1999); supplementary material available at ⟨`http://www-math.mit.edu/~rstan/ec`⟩.

[12] N. J. A. Sloane, *Online Encyclopedia of Integer Sequences*, available at ⟨`http://www.research.att.com/~njas/sequences`⟩.

[13] A. Lobb, Note 83.08: Deriving the n-th Catalan number, *Math. Gaz.*, **83** (1999), 109–110.

[14] E. Swylan, Note 86.62: Proving numbers Catalan, *Math. Gaz.*, **86** (2002), 460–463.

[15] A. Hirst, Roll-a-penny probabilities, *Math. Gaz.*, **73** (1989), 101–106.

[16] R. Hill, Counting and the reflection principle, *Math. Gaz*, **75** (1991), 308–312.

[17] R. H. Cobb, (a) Problems solved and unsolved, *Math. Gaz.*, **60** (1976), 1–10; (b) Problems: comments, corrections and further solutions, *ibid*, **60** (1976), 280–291.

[18] M. Griffiths, 'Catch-up' numbers, *Math. Gaz.*, **91** (2007), 500-509.

[19] L. Zhou, (a) Problem 11187: Tiling 4-rowed rectangles with dominoes, *Amer. Math. Monthly*, **112** (2005), 929; (b) Solutions separately by (i) Northwestern University Math Problem Solving Group; (ii) C. C. Heckman; and (iii) J. Minkus; with editorial comments, *ibid*, **114** (2007), 554–556.

[20] D. G. Rogers, An application of renewal sequences to the dimer problem, *in* A. F. Horadam and W. A. Wallis, eds, *Combinatorial Mathematics VI: Proc. Sixth Austral. Conf., August, 1978. Lecture Notes in Math., 748* (Springer, Berlin, 1979), pp. 143–153.

[21] T. Cross, Student Problem 2003.3, *Math. Gaz.*, **85** (2003), 336; 527–528.

[22] N. I. P. MacKinnon, personal communication, October, 2006.

[23] R. C. Lyness, Al Capone and the death ray, *Math. Gaz.*, **25** (1941), 283–287.

[24] D. F. Lawden, On the solution of linear difference equations, *Math. Gaz.*, **36** (1952), 193–196.

[25] S. Abbott, Note 79.31: A difference method for $\sum m^k p^m$, *Math. Gaz.*, **79** (1995), 355–359.

[26] W. Feller, *An Introduction to the Theory of Probability* (J. Wiley and Sons, New York, NY, 1950; 2nd. ed., 1957; 3rd. ed., 1968).

[27] M. S. Renault, Lost (and found) in translation: André's *actual* method and its application to the generalized ballot problem, *Amer. Math. Monthly*, **115** (2008), 358–363; also available at ⟨http://webspace.ship.edu/msrenault/ballotproblem/⟩.

[28] M. S. Renault, Four proofs of the ballot theorem, *Math. Mag.*, **80** (2007), 345–352.

[29] M. D. Atkinson and W. F. Lunnon, Regular fault-free rectangles, *Math. Gaz.*, **64** (1980), 99–106.