Pythagoras sheared, Euclid dissected: Are they cut out for scissors congruence?

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Dedicated to the New England Mathematical Association (NEMA), on the occasion of its fiftieth anniversary (1958–2008)

I have often been surprized that Mathematics, the Quintessence of Truth, should have found admirers so few and so languid — Frequent consideration and minute scrutiny have at length unravelled the cause — Viz — That, though Reason is feasted Imagination is starved; whilst Reason is luxuriating in its proper Paradise, Imagination is wearily travelling on a dreary desart. To assist Reason *by the stimulus of* Imagination *is the* Design *of the following production.* S. T. Coleridge [16, p. 7], quoted in [10]

1 The shearing dissection of Cundy and Rollett

Mathematical Models[10], by (Henry) Martyn Cundy (1913–2005) and Arthur Percy Rollett (1902–1968), appeared from the Clarendon Press in 1951 and quickly established itself as a well-loved classic, going through three editions and frequent reprinting. Both authors were gifted school teachers of long experience who were to go on to other levels of engagement with mathematics: Rollett as an Inspector of Schools and, in the year before his death, President of the Mathematical Association; Cundy as a Professor in Malawi. Arthur Rollett had taught at Sevenoaks School from 1926 (confirmed in position, 1927) until the mid-1940s, following a degree from East London College (later Queen Mary College), University of London and a brief stint at a county secondary school in Middlesex. Martyn Cundy, after a distinguished period of study at Cambridge, taking a starred Part III in the Mathematical Tripos of 1935 and a Rayleigh Prize two years later, had then opted for teaching at Sherborne School, instead of continuing with university work at that stage. While it was Rollett who provided the initial impetus and ideas for the undertaking, Cundy was responsible for its execution as a finished text. Their purpose was to capture and foster mathematical imagination through the use of models that gave a true feel for the subject — a purpose they cleverly declared in launching their *Preface* with the words of a youthful Samuel Taylor Coleridge (1772–1834), written to his brother George on 31 March, 1791, that we take anew as our own epigraph. This quotation is all the more apposite on recalling that the budding poet was just two months into his time at Jesus College, Cambridge, and grappling with his undergraduate studies, of which he told George early that November [16, p. 16]:

> *We have Mathematical Lectures, once a day — Euclid and Algebra alternately. I read Mathematics three hours a day — by which means I am always considerably before lectures, which are very good ones.*

The materials Cundy and Rollett selected for their chapters are equally "*very good ones*", rebuffing any suggestion that imagination in mathematics need necessarily be starved in a dreary desert. In particular, early in Chapter 2, on models in plane geometry, they present the *dynamic* demonstration of the Pythagorean Proposition shown in Figure 1 [10, §2.1.4, Fig. 10]. Readers of *The Australian Mathematical Society Gazette* have had a recent reminder of this in a review [7, (c)] in 2006 of a book [7, (b)] on *state-of-the-art* illustration for mathematical texts. In a familiar sequence of silent frames, we see the squares on the legs of a right triangle being sheared into parallelograms, and then dropped down, to be trimmed into rectangles partitioning the square on the hypotenuse — '*shear–translate–shear*'. Since areas are preserved at each step, we conclude that the sum of the squares on the legs is equal to the square on the hypotenuse, as an assertion about areas. Although this is little more that a *re-presentation* of the proof of *Elements I.47* in the general form that appears in the *Mathematical Collections* of Pappus (c. 290–c. 350), near the beginning of Book IV, still it makes a pleasing appeal to the imagination (compare [20, (a) §213]; we consider generalisations in our final Section).

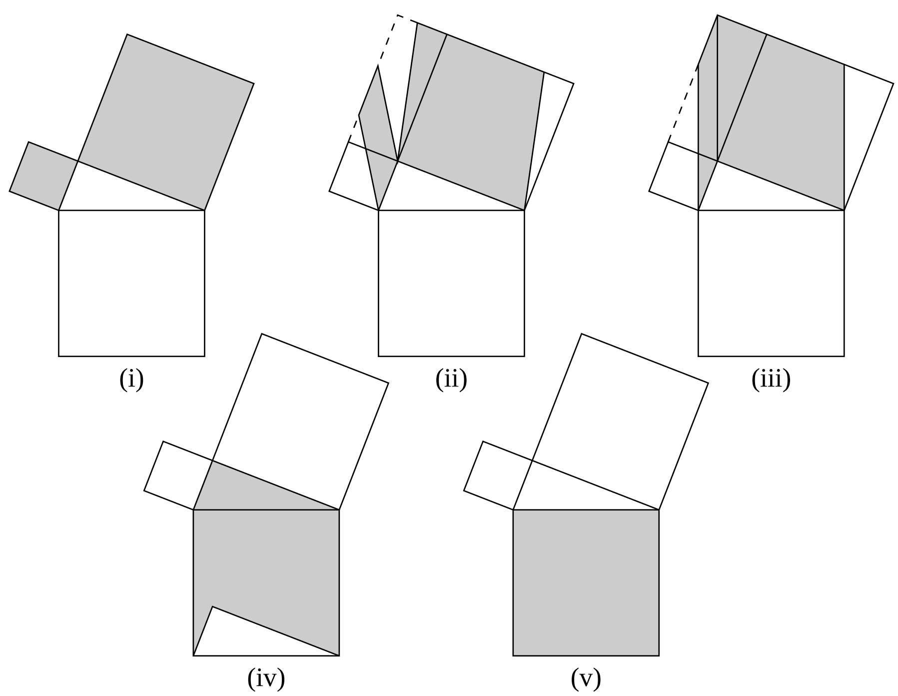


Figure 1: Dynamic demonstration, after Hermann von Baravalle, c. 1926

Effective teachers who appreciate the mechanics of a geometrical argument have long understood how this helps motivate proof, without necessarily being conscious in this regard of the *Erlanger Programm* of Felix Klein (1849–1925). To give just one example bearing on the proof Euclid gives of *Elements I.47*, at a discussion [24,

p. 78] of methods of learning geometrical theorems, held in London in 1933, we find Edith Florence Edwards (1879–1966), Headmistress throughout the 1920s and 30s of Fairfield High School, Droylsden (founded 1796), pointing out that:

> *children were helped to remember the construction for proving Pythagoras' theorem if they were shown that the second triangle was obtained by twisting the first through a right angle.*

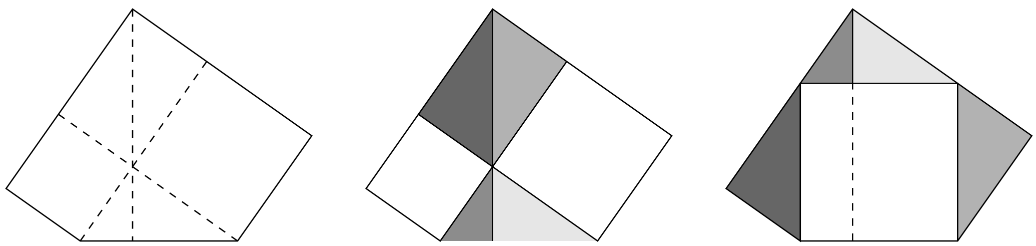
It is certainly a point for instructors to keep in mind when covering Euclid's proof of *I.47*. Sybrandt Hanszoon van Harlingen (Cardinael; 1578–1647) seems to hint at just such a '*shear–rotate–shear*' visualisation, using parallelograms rather than triangles, in a single frame emblazoned on the title page of a problem book from about 1612, except that angles are marked inconsistently with rotation (see [25, Fig. 5]). More recemtly, in [5, p. 187], this depiction is unpacked in a sequence of four unannotated frames, although again perhaps the middle step may not register as a rotation so obviously as do the shearings on either side. But contemporary computer-aided printing can bring such an idea to life, in this case as displayed in [7, (a), Fig. 5, p. 159], more fully and flexibly than was possible even a dozen years ago, not to mention a lifetime. Now we shall have no excuse for missing the point.

Nonetheless, long before these modern advantages, Hermann von Baravalle (1898–1973), one of the founding complement of teachers at the original Steiner-Waldorf school in Stuttgart in the 1920s, devised teaching materials from which a version of Figure 1 evolved as part of a portfolio of '*geometry in pictures*' [3, (a)] issued in 1926. He republished this in more definitive form [3, (b, c)] in the mid-1940s for a new audience, having emigrated to the USA in 1937. It received a good press, being taken up by E. T. Bell (1883–1960) in [4] in 1951, as well as by Martin Gardner in [14] in 1964, coming thereby to enjoy wide currency. While Cundy and Rollett, unlike Bell and Gardner, do not refer to Baravalle by name, it seems plausible that his work also inspired their use of Figure 1, in which, however, they make the helpful innovation of reversing the order of the frames, so as to start more naturally with the squares on the legs of the right triangle. But Baravalle's standing as a mathematician, and even as a mathematics educator, is today less recognised, being if anything rather eclipsed by his contributions to Steiner-Waldorf education more generally — he is not mentioned in [7, (a, b)] nor, for that matter, in the stories retold in [13, (a), Fig. 17, p. 75; (b), Fig. 9, p. 33], despite his community of outlook with these authors.

Now, having presented Figure 1, Cundy and Rollett make the throw-away observation that, were this shearing demonstration of *Elements I.47* to be encapsulated by means of a dissection, it would require eight pieces. Readers who have prized *Mathematical Models*, especially those of a practical disposition, might be expected to rise to the challenge implicit in such a remark. So, it has certainly shown staying power to have survived through three editions and several reprintings. Of course, it is clear that the dissection of the square on the longer leg can be accomplished in four pieces to simulate shearing (compare Figure 9). Symmetry might suggest that the square on the shorter leg can be dissected similarly in another four pieces, for a total of eight. In some cases, that is true. But there is more to this than meets the casual eye; and we hope that readers will find a second glance as instructive as we have.

2 Dissecting *Elements I.35*

The dynamic demonstration of *Elements I.47* can be caught as it were *in silhouette* using three congruent right triangles on a pentagonal board, with one of the right triangles divided along the altitude perpendicular to the hypotenuse (see Figure 2). In one setting (Figure 2(ii)) of the four moveable pieces, the squares on the legs are left vacant, while translating the pieces on the board into a second setting (Figure 2(iii)) reveals the square on the hypotenuse (compare [20, (a) §210], echoed in [20, (b) §233]). While this approach to *I.47* goes back at least to al-Sabi Thâbit ibn Qurra al-Harrani (836–901), it would seem not to be the sort of dissection that Cundy and Rollett had in mind. However, Figure 2 does remind us that Euclid, in working towards the proof of *Elements I.47*, did not prove *I.35*, that parallelograms on the same base and between the same parallels have equal area, by dissecting one parallelogram into pieces that could then be reassembled into a second parallelogram.



(i) board (ii) squares on legs (iii) square on hypotenuse

Figure 2: Equicomplementable shearing

Indeed, one common proof of *Elements I.35*, the "*window shutter*", shown in Figure 3, resembles Figure 2, in that the demonstration is effected by putting the shutter in two different positions to leave "*open*" one or other of the parallelograms. But this is from the improving hand of some latter-day commentator, and is not the work of Euclid, For, Euclid's proof of *I.35* has been grist to those, like Proclus Diadochus (411–485), bent on distinguishing cases.

(i) (ii)

Figure 3: *Elements* I.35 — *the window shutter*

As it happens, this is unnecessary here, since Euclid's proof works quite generally, in so far as what he does is acceptable in terms of what has gone before in *Elements I*. Thus, we have parallelograms $ABCD$ and $EBCF$, on common base BC and with A, D, E and F on a line parallel to BC. Without loss of generality, let us agree that, on reading from left to right along that parallel line, it is A that is encountered first of all. Then, if the parallelograms are distinct, BE and CD are not parallel, so meet in some point, G, say. At this juncture in *Elements I*, all that Euclid has at his disposal as regards areas is the congruence of triangles. Consequently, Euclid offers decompositions of the parallelograms in terms of the juxtaposition (addition) or cutting off (subtraction) of contiguous triangles:

$$ABCD = \Delta EAB + \Delta CBG - \Delta DEG$$

and

$$EBCF = \Delta FDC + \Delta CBG - \Delta DEG.$$

The only modification we have made to Euclid's argument is to write these decompositions so as to ensure that an area is always present before it is cut off, to allow for the possibility that G may not be between the parallel lines (the case traditionally illustrated; compare Figure 4(i) and (ii)). But triangles ΔEAB and ΔFDC are congruent. So, Euclid concludes from his decompositions that the two parallelograms are equal in area, as was required to be demonstrated.

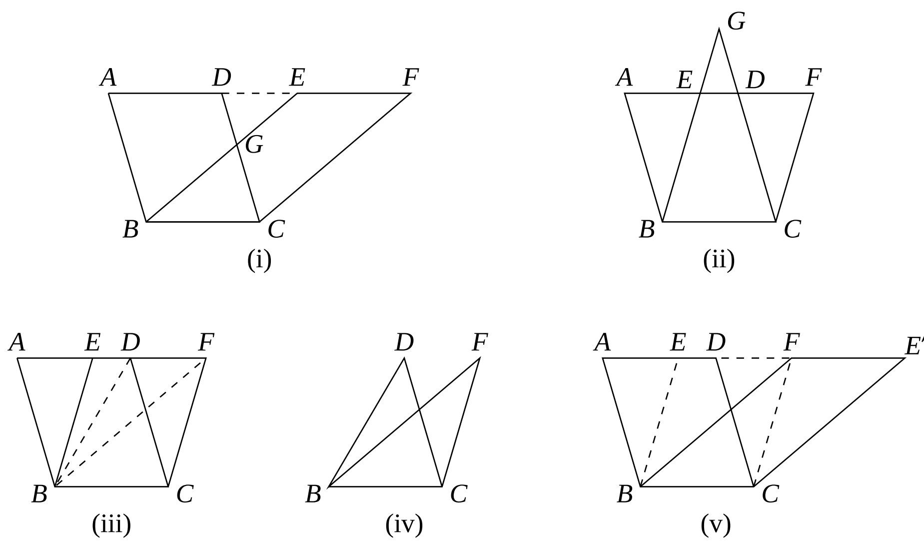


Figure 4: *Elements* I.35, 37 — after Euclid

If it still be pressed that something needs to be said separately about the cases where AD and EF do or do not overlap, then consideration of *Elements I.37*, that *triangles* on the same base and between the same parallels have equal area, helps clarify the picture. For, if AD and EF intersect, as in Figure 4(iii), then the triangles ΔBCD and ΔBCF, shown in Figure 4(iv), are exactly as envisaged in *I.37*. However, Euclid does not move back to Figure 4(iii), but rather on to Figure 4(v), in which parallelograms $ABCD$ and $FBCE'$, in addition to being on the same base and between the same parallels, have AB and FE' disjoint. So, even if there are two cases to be distinguished, Euclid is aware that one can be transformed into the other — of course, the trick here is that triangles $\Delta FEB, \Delta BCF$ and $\Delta E'FC$ are all congruent.

Readers of the *Gazette* will be attuned by [11, (b)] to the unexpected subtleties to be found in such fundamental notions as area and volume. One natural approach to area is by dissection: two regions are said to be *equidecomposable* if they can be dissected into the same, finite number of subregions, typically triangles, such that there is a bijection between the subregions in one dissection and congruent subregions in the other (the subregions here must clearly be such that *congruency* itself is well-defined, as it is for triangles). A classic result, under the presence of an Archimedean axiom, such as Euclid introduces later as *Definition 4* in *Elements V*, is that rectilinear regions have the same area if and only if they are equidecomposable. As in [11, (b) p. 83], this theorem is traditionally attributed to Wolfgang Farkas Bolyai (1775–1856) and P. Karl Ludwig Gerwien (1779–1858), writing in the early 1830s. But it can be found in work of William Wallace (1768–1843) up to a quarter of a century earlier, as well as in a contribution by John Lowry (1768–1850) in 1814 to a question raised by Wallace (see [18] for recent discussion; that such issues exercised some general, if non-technical, interest in that period can be seen from the chapter devoted to them in the popular manual [20, (b) Ch. IX, pp. 93–110] of 1831). In this terminology, Euclid's proof of *Elements I.35* shows that the parallelograms $ABCD$ and $EBCF$, each with the triangle ΔDEG adjoined, are equidecomposable.

But this, in turn, suggests a second approach to area: two regions are *equicomplementable* if they can be augmented to equidecomposable regions by adjoining to each the same finite number of disjoint subregions, such that the subregions augmenting one region are in one-to-one correspondence with congruent subregions augmenting the other. Thus, at the risk of making Euclid sound like a character out of Molière, we can now rephrase *I.35* to say that the parallelograms $ABCD$ and $EBCF$ are equicomplementable, as certified by the decompositions Euclid presents for them in his proof. Of course, we expect that pairs of regions that are equicomplementable are also equidecomposable and *vice versa*. This was confirmed in a more general context by Jean-Pierre Sydler (1921-1988) in [26]. However, Euclid tends to use whatever comes most conveniently to hand — in *I.45* and, more conspicuously, in *I.47*, he even runs ahead of himself in fusing rectangles with a common side into a larger rectangle, something he does not cover officially until *II.2*.

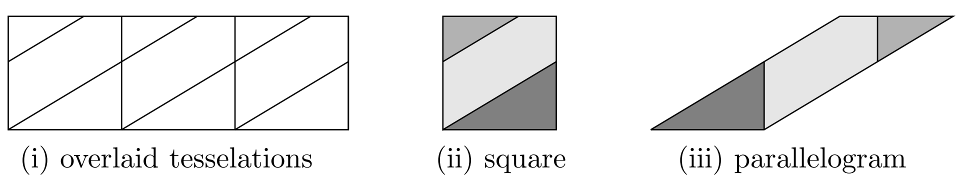
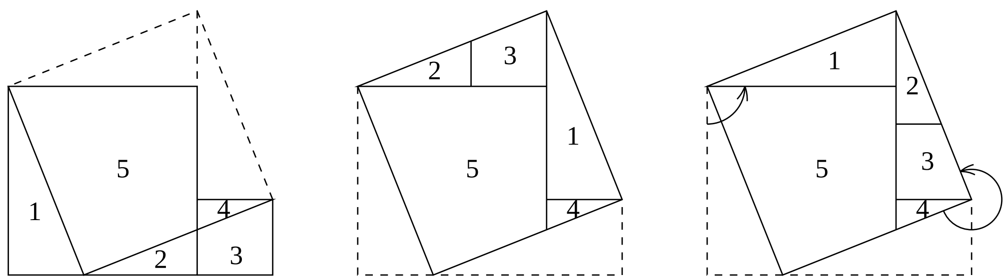


Figure 5: Dissecting a square into a parallelogram

The need for an Archimedean axiom soon makes itself felt in any attempt to implement Cundy and Rollett's suggestion to mimic Figure 1 by dissecting a square into a parallelogram or, more particularly, into a rectangle, of equal area. The challenge here is to confirm directly that they are indeed equidecomposable, rather than to infer this, following Euclid, as a consequence of showing that they are equicomplementable. A simple enough method of dissection is by means of tessellations: as in Figure 5, we overlay a mesh of congruent parallelograms on a uniform square grid, and then sort out the pieces. But it is immediately clear that the number of subregions will be determined by how many squares in the grid a given parallelogram straddles. In terms of a right triangle with legs a and b, where $a \leq b$, we find ourselves needing to know that there is an integer n such that $na < b \leq (n+1)a$. Little surprise then that Euclid went another route in proving *Elements I.35*.

3 A *Euclidean* dissection for *Elements I.47*

Thâbit ibn Qurra seems to have had some sense of the distinction made formally only much later between regions being equidecomposable and their being equicomplementable. His demonstration of *Elements I.47* caught in Figure 2 is one in which regions, on the one hand the squares on the legs of a right triangle, and, on the other, the square on that triangle's hypotenuse, are shown to be equicomplementable. But Thâbit ibn Qurra also established that these regions are equidecomposable by means of the dissection shown in Figure 6, thereby providing an alternative demonstration of *I.47*.

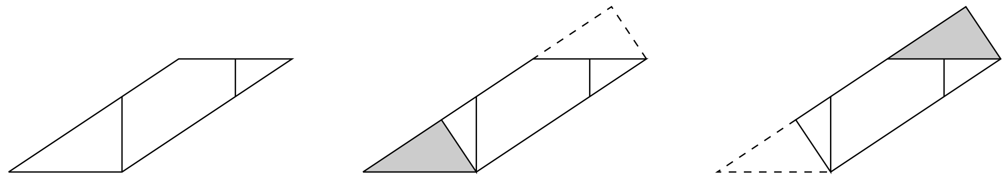


(i) side-by-side (ii) by translations (iii) by rotations

Figure 6: Direct dissection demonstration of *Elements* I.47

For this direct dissection demonstration of *I.47*, as we see in Figure 6(i), the squares on the legs of a right triangle are aligned side-by-side, with two copies of the right triangle cut from them. The resulting five pieces (subregions) can then be rearranged to form the square on the hypotenuse (compare [20, (a) §212]). However, the mechanics of this transformation can be viewed in *two* ways, although some authors seem to favour one to the exclusion of the other. In one view, as in Figure 6(ii), the copies of the right triangle are translated across the square on the hypotenuse, from outside to inside. In the other, as in Figure 6(iii), they are rotated about opposite corners of the square on the hypotenuse. That the translational version is minimal in a certain logical sense was shown by (Rudolf Hermann) Hans Brandes (1883–1965) in a dissertation [6] in 1907 (see [22] for further discussion of this result and [15] for information on Brandes; the comparable status of the rotational version and of Thâbit ibn Qurra's equicomplementability demonstration seems not to have been investigated).

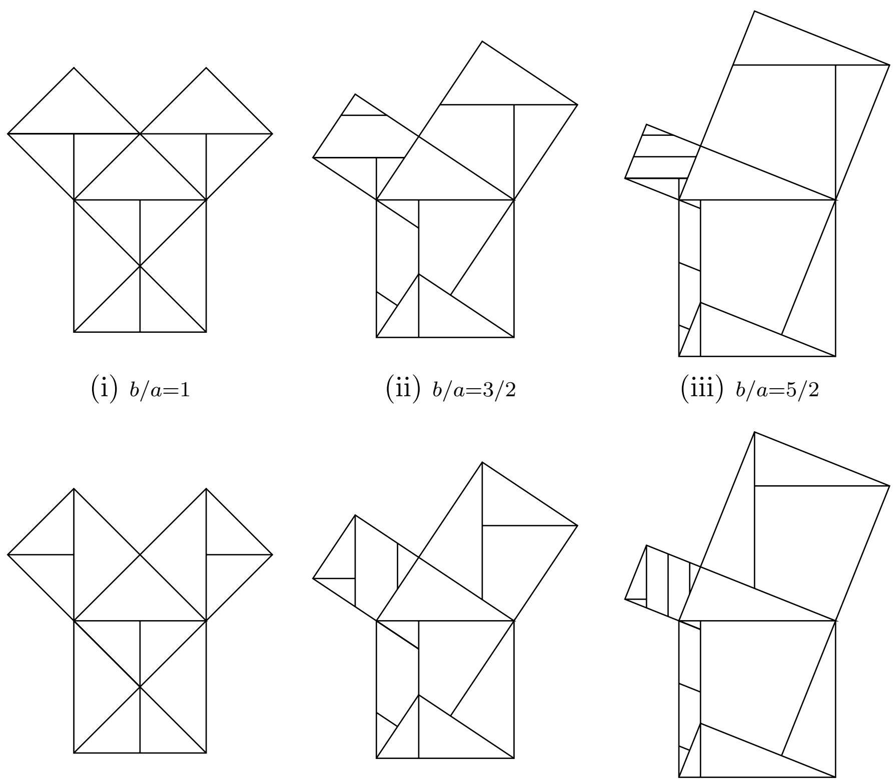
Heinz Hopf (1894–1971), in a celebrated lecture course [17, p. 64] at Stanford University in 1946, enlarged upon this minimality result. Hopf noted that Euclid's own strategy in proving *Elements I.47* converted the square on each leg separately into rectangles having an edge in common that together form the square on the hypotenuse, but that the dissection in Figure 6 does not respect this *Euclidean division*. Since the theory of equidecomposability covered in Hopf's lectures guarantees that the squares on the legs can, in fact, be dissected into rectangles dividing the square on the hypotenuse after the manner of Euclid, Hopf posed the question [17, p. 75] of finding such dissections, if possible with a minimal number of pieces.



(i) dissected parallelogram (ii) uniform trim (iii) dissected rectangle

Figure 7: Trimming parallelogram into rectangle — uniform trim

Leaving aside the issue of minimality, to answer Hopf's question it is enough to trim a dissected parallelogram obtained by superimposed tessellations, as in Figure 5, to produce a rectangle, the simplest trimming cut being the uniform one shown in Figure 7. Indeed, this is exactly what is adopted by Dionysius Lardner (1793–1859) in *A Treatise on Geometry, and its Application in the Arts* [20, (b)], a popular tract that appeared in 1831 — it is reputedly the first geometry book to bring paper-folding to the aid of explicating an argument [20, (b) §10]. Lardner had already shown his expository skill in an edition [20, (a)] of the first six books of Euclid's *Elements* published in 1828 for the use of students in and preparing for the newly formed University College, London, where he had been appointed foundation Professor of Natural Philosophy and Astronomy the previous year.



(i) $b/a{=}1$ (ii) $b/a{=}3/2$ (iii) $b/a{=}5/2$

(iv) $b/a{=}1$ (v) $b/a{=}3/2$ (vi) $b/a{=}5/2$

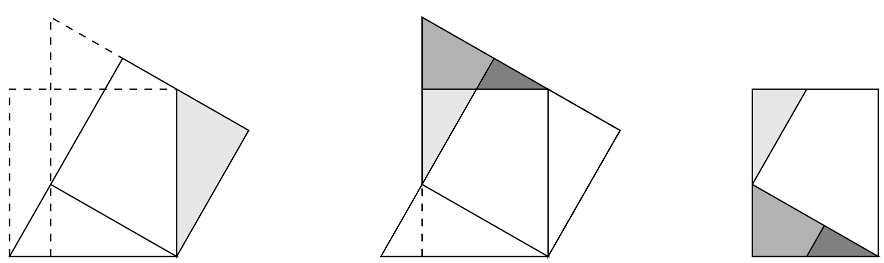
Figure 8: Lardner's *Euclidean* dissection for I.47

A merit of this edition is that Lardner provides supplementary observations, including, when it comes to *Elements I.47*, arguments [20, (a) §§210, 212] on the lines of both Figure 2 and Figure 6. The former reappears in [20, (b) §233, Figs. 108, 109]. But, in this more demotic work, he prefaces it with a dissection demonstration [20, (b) §233, Fig. 107] of *I.47* that adheres strictly to the scheme of Euclid's proof. For, not only does it respect the division of the square on the hypotenuse by the perpendicular altitude of the right triangle, but the pieces are rotated into position, as shown, in some typical cases, in Figures 8(i)–(iii). However, it is easy to modify these dissections to effect the transformation by translations (see Figures 8(iv)–(vi)).

For a right triangle with legs a and b, where $a \leq b$ this style of dissection requires three pieces for each square when $a = b$, but, if $0 < na < b \leq (n+1)a$, the square on the shorter legs requires $n + 3$ pieces, although that on the longer leg still requires only three pieces. As it happens, Cundy and Rollett, in their version [10, §2.1.4, Fig. 10] of Figure 1, do seem to illustrate a case where $n = 2$, for which a dissection of this type uses eight pieces. But that seems to be a happy accident, since, whatever else Lardner's dissections do, they do not mimic the shearing action that motivates Figure 1, so it would not appear to be of the sort Cundy and Rollett meant. To capture that requires a different trimming cut, to which we now turn.

4 A *shearing* dissection for *Elements I.47*

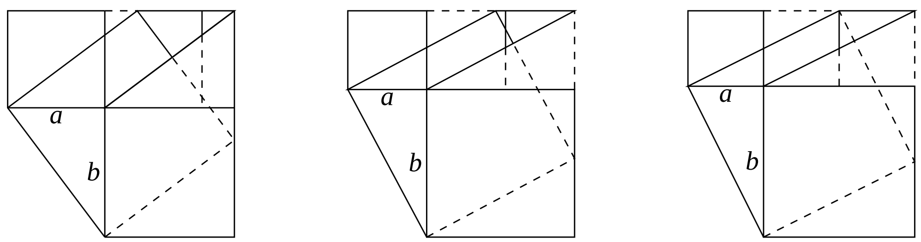
It is apparent, from Figures 1(iii) and 2(iii), that if a square on the hypotenuse of a right triangle overlaps the triangle and squares placed externally on the legs, then it cuts off a congruent right triangle from the square on the longer leg. The shearing of the latter square into the shaded parallelogram in Figure 1(iii) can be simulated as a dissection by translating this triangle from inside the square, as in Figure 9(i), to outside, as in Figure 9(ii). The trimming cut needed to convert the parallelogram to a rectangle passes through both sections of the square to give, in Figure 9(iii), a dissection of the rectangle from the square in four pieces, as foreshadowed at the close of Section 1. Thus, this shearing dissection for the square on the longer leg is effected using one more piece than required for Lardner's dissections in Figure 8.



(i) shearing (ii) trimming (iii) rectangle

Figure 9: Shearing dissection for square on longer leg

The square placed externally on the shorter leg of the right triangle overlaps the corresponding sheared parallelogram in a similar right triangle, as can be seen in Figure 1(iii). The key to implementing a shearing dissection in this case is that the overlapping region remains intact as a piece in the dissection. Consequently, the uniform trim used in Lardner's dissections is unavailable, and the sheared parallelogram has to be trimmed from the other end. But then the number of regions cut exhibits a more subtle dependency on the ratio between the legs of the right triangle. For example, considering a right triangle with legs a and b, where $a < b \leq 2a$, Figure 10 indicates that early in this interval and again at the end, the dissection uses four pieces, while there is a portion of the interval towards the end where this number rises to five.



(i) $b/a{=}4/3$ (ii) $b/a{=}15/8$ (iii) $b/a{=}2$

Figure 10: Catching the first transition

Naturally, interest now centres on determining where in the interval this transition in the shearing dissection takes place. To this end, let ΔABC be the right triangle with the square $ABKH$ on the hypotenuse AB placed over it, but the square $BDEC$ on the shorter leg CB placed externally (see Figure 11).

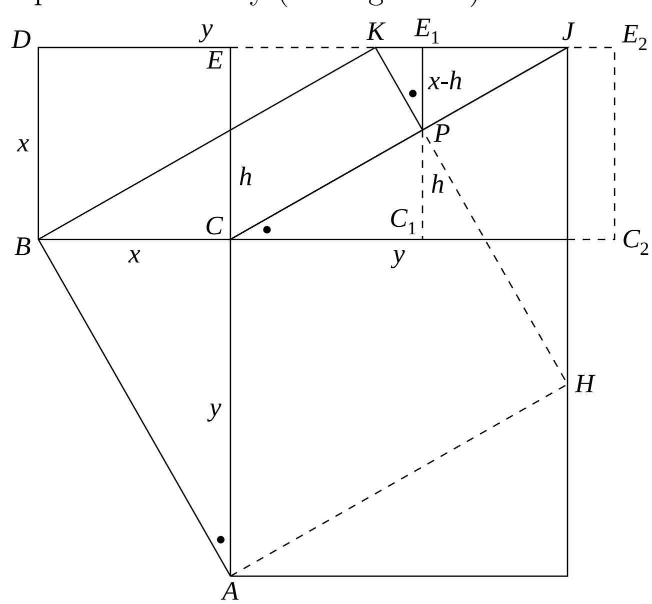


Figure 11: Locating the first transition point

Further, let ΔHKJ be congruent to ΔABC; and let the square CEE_1C_1 be congruent to $BDEC$. Then the largest ratio of the legs CA and CB with $1 < CA/CB < 2$ for which the shearing dissection can be effected in only four pieces occurs when, referring to Figure 11, HK, CJ and C_1E_1 meet in a point, P, say. In this case, $\Delta BAC, \Delta PCC_1$ and ΔKPE_1 are all similar, so that

$$\frac{CA}{CB} = \frac{C_1C}{C_1P} = \frac{E_1P}{E_1K}. \tag{1}$$

Thus, in the notation in Figure 11,

$$\frac{y}{x} = \frac{x}{h} = \frac{x-h}{2x-y}, \tag{2}$$

observing that $2x - y > 0$ in the interval under consideration. It follows, on writing $t_1 = y/x$ and eliminating h, that t_1 is the root of the cubic equation

$$t^3 - 2t^2 + t - 1 = 0, \tag{3}$$

in the interval $1 < t < 2$; indeed, from Figure 10, $4/3 < t_1 < 15/8$. We conclude that, for a right triangle with legs a and b, where $a < b \le 2a$, the number of pieces in the shearing dissection is four when $1 < b/a \le t_1$ or $b = 2a$ and five otherwise.

For a right triangle with legs a and b, where now $a < na < b \le (n+1)a$, the story is much the same. In an obvious notation extending that of Figure 11, we have, on analogy with (1) and (2):

$$\frac{CA}{CB} = \frac{C_nC}{C_nP} = \frac{E_nP}{E_nK}; \qquad \frac{y}{x} = \frac{nx}{h} = \frac{x-h}{(n+1)a-b}.$$

So, in this interval, the shearing dissection has $n+3$ pieces when $n < b/a \le t_n$ or $b/a = n = 1$, and $n+4$ pieces when $t_n < b/a < n+1$, where t_n is the root of the cubic equation

$$t^3 - (n+1)t^2 + t - n = 0, \tag{4}$$

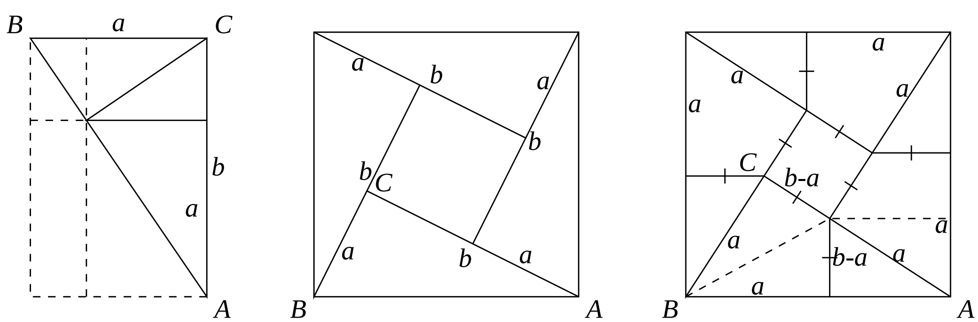
with $n < t < n+1$. Moreover, a little algebra reveals that

$$(t_{n-1}+1)^3 - (n+1)(t_{n-1}+1)^2 + (t_{n-1}+1) - n = (2t_{n-1}+1)(t_{n-1}-n) < 0, \quad n > 1.$$

So, rather more precisely,

$$t_{n-1} + 1 < t_n < n+1;$$

and $n + 1 - t_n$ tends to zero as n increases. As in Figure 10, it is interesting to probe this convergence by means of right triangles with integer sides. For example, $t_2 < 35/12 < 3$, but $63/16 < t_3$. In this spirit, for those who enjoy the number theory of Pell equations, it may be an amusing exercise to find right triangles with integer legs a and $b = (n+1)a - 1$ giving integer hypotenuse and $t_n < b/a < n+1$.



(i) Supergolden (ii) Abu'l-Wafa (iii) Omar Khayyam

Figure 12: Right triangles with special properties

The occurrence of cubic equations in (3) and (4) is a reminder, if any were needed, of the extensive presence of such equations in geometrical settings, often related to right triangles — some other instances appear in [8] and in a further contribution [9] on the snub-cube from Martyn Cundy in his late eighties. The problem of the *supergolden rectangle* in [8] can, for example, be recast as that of finding a right triangle with legs a and b, where $a < b$, such that the longer segment cut off the longer leg by the perpendicular from the foot of the altitude on the hypotenuse is equal to the shorter leg, as illustrated in Figure 12(i) (compare [8, Fig. 4]). Setting $u = b/a$, then, as shown in [8],

$$u^3 - u^2 - 1 = 0.$$

Problems of this sort have a long history. It was another example that motivated Omar Khayyam (1048–1122) to study and classify cubic equations — Omar Khayyam's paper is translated in [1], but the late Alpay Özdural (1944–2003) retold the story of Omar Khayyam's researches from the perspective of an architectural historian in an engaging series of articles [21] that explores the possible interplay between geometer and artisan. The familiar arrangement in Figure 12(ii) of four congruent right triangles right triangles, set internally in a square on their hypotenuses to form an inner square, is much of a piece with Figures 2 and 6, as yielding yet another visual demonstration of *Elements I.47*. Not only was this known to Mohammad Abu'l-Wafa Al-Buzjani (940–998), but he saw how, reflected along the edges of the containing square, it forms a pleasing geometrical pattern of squares and right kites that underlies many ancient Arabic designs. Moreover, one way to enhance this pattern is to extract a right kite from each of the right triangles, as shown in Figure 12(iii). Omar Khayyam observed that what is required is a right triangle with hypotenuse equal to the altitude perpendicular to it together with the shorter leg a. An equivalent condition is that the longer leg b is equal to the shorter leg a together with the shorter segment cut off the hypotenuse by the perpendicular altitude. He then deduced, in effect, that the ratio $v = b/a$ satisfies the cubic equation

$$v^3 - 2v^2 + 2v - 2v = 0.$$

But he also sensed that it would not be possible to find such a right triangle by Euclidean means of straightedge and compasses alone, offering instead a solution as the intersection of a parabola and a hyperbola. As it happens, solutions to all these problems leading to cubic equations can be obtained by paper-folding. Construction methods may, of course, fail, but, as observed in [11, (a) §4], there are more general results on links between tilings with similar figures and irreducible polynomials with integer coefficients.

We are pleased to think that Cundy and Rollett would have enjoyed finding that their proposed shearing dissection for *Elements I.47* had both a greater intricacy and a richer historical resonance than they might have supposed in writing [10], especially in view of Cundy's late interest [9] in the cubic

$$w^3 - w^2 - w - 1 = 0.$$

But it remains something of a mystery why they thought it could be accomplished in eight pieces without further qualification, all the more since earlier in their text they are alert to constraints in dissecting a triangle into a rectangle (see [10, Figs. 4, 5] and compare [11, (b) p. 82]).

5 A dynamic approach to the Law of Cosines

We comment briefly on the Law of Cosines — *Elements II.12,13* as presented by Euclid — only because it seems sometimes suggested that the dynamic approach of Hermann von Baravalle does not carry over to general triangles, just as a similar question arises with Euclid's proof of *I.47*. Euclid distinguishes the cases of obtuse angles, in *II.12*, and acute angles, in *II.13*, because of a predilection in *Elements II*, as manifest in earlier pairings of propositions — *II.4* with *II.7*, *II.5* with *II.6* and *II.9* with *II.10* — for juxtaposition (addition) of line segments to the exclusion of overlapping (subtracting) them, in contrast to the treatment of areas in the proof of *I.35*, as discussed in Section 2, where avoiding subtraction would be more difficult. The question at issue in *II.12, 13* is where the foot of the perpendicular from the vertex containing the angle falls externally or internally on the opposite side of the triangle. Similarly, in generalising the proof of *I.47*, interest centres in the location of the *orthocentre* of the triangle. A triangle can have one internal altitude, in which case the orthocentre lies outside the triangle, as in Figure 13(i), or all three altitudes internal, meeting therefore inside the triangle, as in Figure 13(ii), with the right triangle a borderline case in which two of the altitudes become sides and the orthocentre is the vertex containing the right angle. Expositors [12, 5] tend to favour an internal orthocentre, as it makes for the more compact diagram, Figure 13(ii). With this generalisation in view, it is only a little more difficult, if somewhat more elaborate, to provide complementary illustrations, in Figures 13(iii) and (iv), on which to base corresponding generalisations of the dynamic demonstration in Figure 1. These latter diagrams have the merit of bringing the yet more general configuration presented by Pappus into closer focus. But, in point of making this type of shearing argument for *I.48*, the converse of *I.47*, such greater elaboration can be circumvented.

Perhaps Euclid himself is a source of confusion here. Proclus tells us that, not only is the extension of *Elements I.47* offered in *VI.31* due to Euclid, but the proof of *I.47* is also original with him. Yet, Euclid conspicuously fails to use his proof technique to capture *II.12, 13*, as well as *I.48* in one fell swoop. Instead, so far as has been traced, the generalisation of Euclid's proof had to await the appearance, in 1647, of *Opus Geometricum* [23, Book I, Pt. 2.44, 45], the major work of Gregorius a Sancto Vincentio (Grégoire de Saint-Vincent; 1584–1667), with another wave of interest stemming from [20, (a) §214] only in 1828. For some, going back at least to Petrus Ramus (Pierre de la Ramée; 1515–1572), discrepancies of this sort point to *Elements* being a composite work of many hands, not always carefully edited together or, as argued for example in [2, §15.3], being rewritten in a different order informed by some other point of view. In particular, much is made of peculiarities regarding *II.12, 13* in [19]. Certainly, a more rudimentary version of *II.12, 13*, in the form of inequalities, can be proved by a simple dissection argument based on *I.47*, in effect giving an alternative proof of *I.48*. Euclid may have wanted to point out, in the proofs he supplies for *II.12, 13*, that *I.47* was sufficiently powerful to establish the law of cosines as exact equalities. Moreover, it is not unknown for instructors to vary their proof techniques for pedagogical reasons.

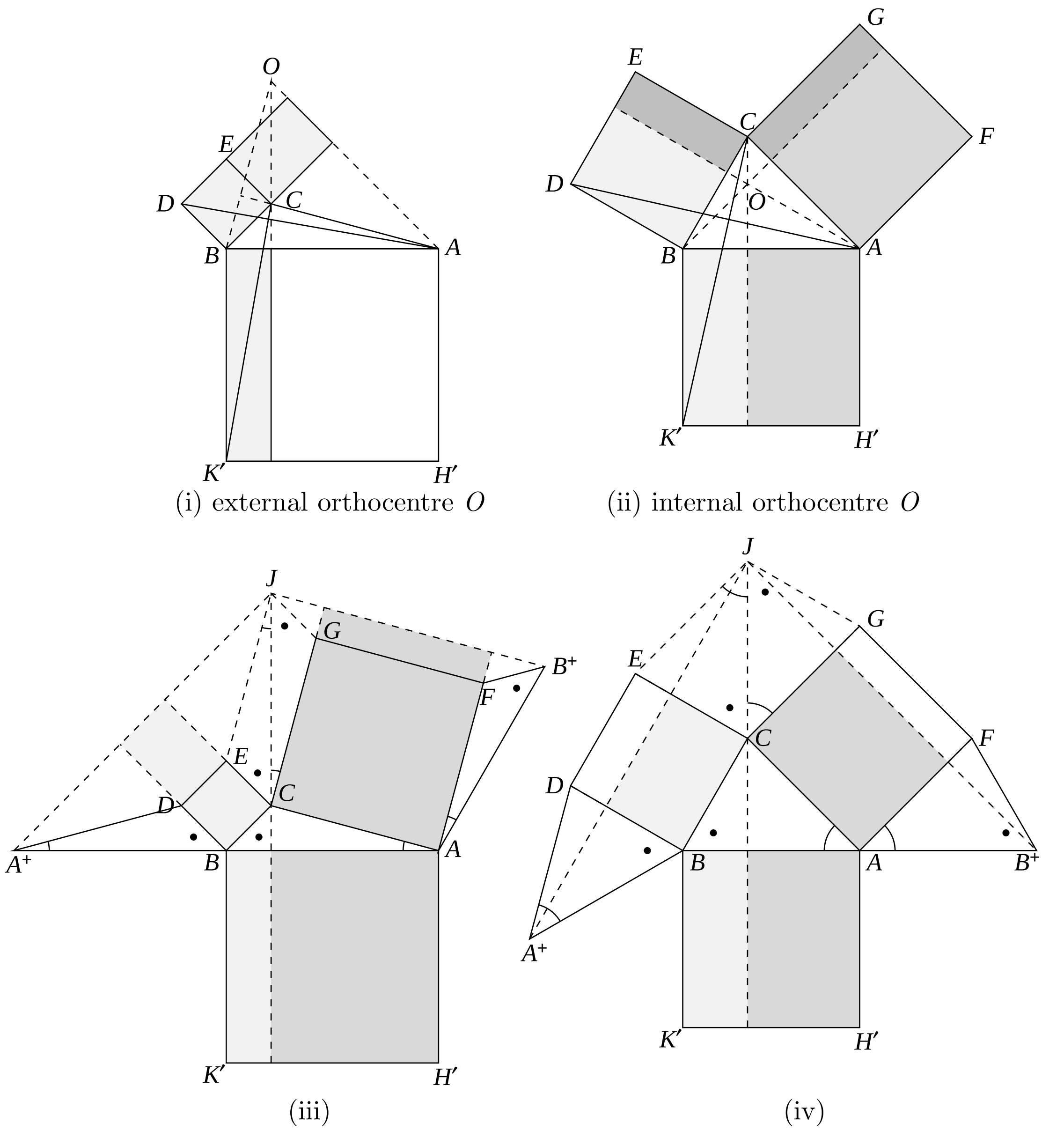


Figure 13: General triangles

But looking at Figure 13, the diagrams themselves, while making a simple point, are more complicated than Euclid needs for his proof, rendering their generality of questionable advantage. After all, such issues seem much a matter of taste even today, as seen on juxtaposing [13, (a), p. 65; (b), p. 37] with [2, §16.9, p. 154].

> *But perhaps the most remarkable extension of the Pythagorean Theorem that dates back to the days of Greek antiquity is that given by Pappus of Alexandria at the start of Book IV of his* Mathematical Collection. [13]

> *There is another generalisation of this due to Pappus … This is trivially true. I have never seen any application of it. It is what one might call a shallow generalization. It is so general that it is no longer of any use.* [2]

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