Bizarre Pool Shots Spiral to Infinity

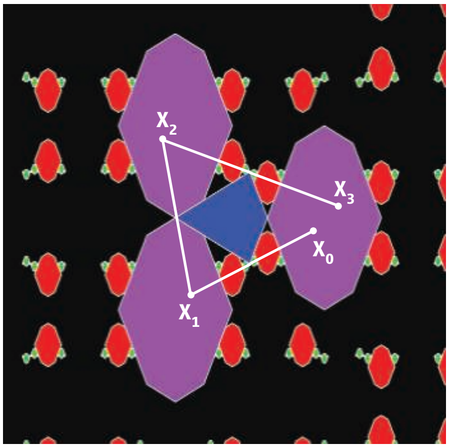
If a mathematician invites you to play billiards, watch out. You're likely to wind up trying to make shots on a table of some weird, polygonal shape—or even on the outside of such a table.

The notion of "outer billiards" was proposed in the 1950s by Bernhard Neumann and popularized (among mathematicians and mathematically minded physicists) in the 1970s by Jürgen Moser as a stripped-down "toy" model of planetary motion. The setup is simple: An object starting at a point x_0 outside some convex figure such as a polygon zips along a straight line just touching the figure to a new point x_1 at the same distance from the point of contact (see figure). It then repeats this over and over, thereby orbiting the figure in, say, a clockwise fashion. Neumann asked whether such a trajectory could be unbounded; that is, whether the object could wind up landing progressively farther and farther from the central figure. This is analogous to the question of whether planetary orbits in the solar system are stable. All proven results, however, went the other way. For regular polygons, all trajectories are bounded, and for polygons whose vertices have rational coordinates, trajectories are not only bounded but also periodic: After a finite number of steps, each trajectory winds up back where it started.

Richard Schwartz of Brown University has given a positive answer to Neumann's question: There is indeed a convex figure with an unbounded trajectory—an infinite number of them, in fact. The example turns out to involve a famous shape, the Penrose kite, which Roger Penrose introduced in the 1970s as one of two pieces (the other is known as the Penrose dart) that produce nonperiodic tilings of the plane with local fivefold symmetry.

Schwartz discovered the unbounded trajectory around the Penrose kite by writing a graphics program for systematically exploring trajectories around kites, which he picked as the simplest figures for which unbounded trajectories could possibly exist. "I think of myself as a good experimenter," he says. "I tried lots of things that didn't work out!"

A key to the discovery was that he computed not only individual trajectories but also entire regions consisting of equivalent trajectories. For the Penrose kite, he found three large, octagonal regions within which trajectories bounce periodically from one region to the other (see figure, below). Around these regions lies a cloud of smaller regions (color-coded red in figure) with similar trajectory behavior, and around these regions is a larger cloud of yet smaller regions, and so on. The larger and larger clouds of smaller and smaller regions, Schwarz found, converged to a set of points from which the trajectories are unbounded.



Outer limits. Billiard balls aimed around a Penrose kite (blue) will travel outward forever, if you pick the right starting point.

Schwartz's initial proof was heavily computational. He has made much of it conceptual, but parts are still computer-assisted. (Schwartz's program, Billiard King, is available at his Web site, www.math.brown.edu/~res.) At the same time, he has found a general class of kites for which, with the help of the computer, he can show unbounded trajectories exist. "The work is very beautiful," says Sergei Tabachnikov, a (mathematical) billiards expert at Pennsylvania State University in State College. "It is an elegant piece of programming and a deep insight into the complicated dynamical phenomena revealed by the experiments." Schwartz, however, admits that the problem is still a puzzlement: "I don't completely understand what's going on."

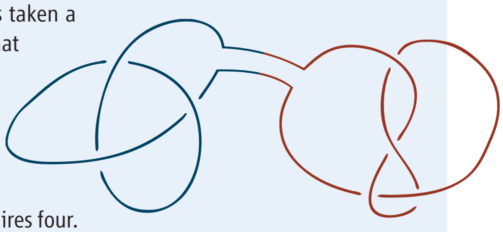
–BARRY CIPRA

That's Not Some Knot Sum!

Knot theory is full of simple-sounding questions that have resisted mathematicians' efforts to answer them for decades. One of the simplest has to do with the minimal number of times a knot has to cross itself when you draw it in two dimensions. In particular, if two knots are strung together to form one larger, more complicated knot (see figure), can the new knot be redrawn with fewer crossings than the original two knots combined?

"This problem has been out there forever," says knot theorist Colin Adams of Williams College in Williamstown, Massachusetts. "It's the most obvious question to ask."

Mathematicians think the answer is no, but the problem has remained stubbornly unsolved. Now, however, Marc Lackenby of Oxford University has taken a small step in the right direction. He has shown that the number of crossings cannot decrease by more than a constant factor—281, to be exact.



Knot theorists denote the minimal crossing number of a knot K by the expression c(K). The trefoil knot, for example, can be drawn with just three crossings, whereas the figure-eight knot requires four. When knots K_1 and K_2 are strung together to form a knot sum, denoted $K_1\#K_2$, the crossing number, $c(K_1\#K_2)$, is obviously no larger than $c(K_1) + c(K_2)$. The conjecture is that $c(K_1\#K_2)$ equals $c(K_1) + c(K_2)$. That is indeed true for the trefoil knot, the figure-eight knot, and all other cases knot theorists have been able to check. But the verification gets unwieldy as the number of crossings increases. It's altogether possible, Lackenby notes, that two knots, each requiring 100 crossings, could be put together and then redrawn with just 199 crossings.

Lackenby's recent result, which he began working on about a year ago, is that $c(K_1\#K_2)$ has to be at least as large as $(c(K_1) + c(K_2))/281$. The basic idea is to think of each knot as enclosed in a spherical bubble and then carefully analyze what must happen to the bubbles if the knot sum is twisted into a new shape with fewer crossings. The analysis produces the factor 281.

To prove the full conjecture, mathematicians need to whittle the number all the way down to 1. Some other approach will be needed for that effort, Lackenby says. "The number [281] is painful to work out," he notes. "One probably can reduce it further, maybe to around 100, but I'm not sure it's worth the effort."

CREDITS (TOP TO BOTTOM): CHAN-HO SUH; RICK SCHWARTZ

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