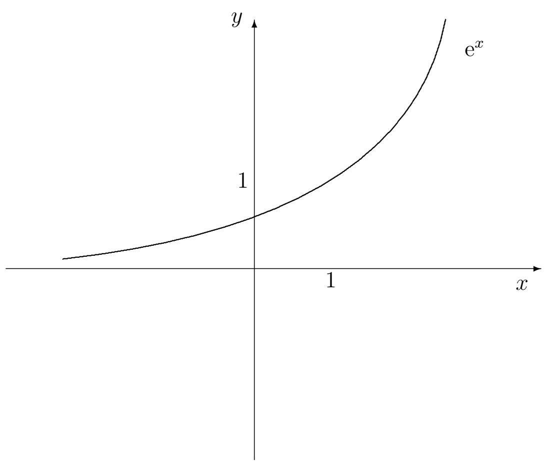
Worksheet 4.9 Exponentials and Logarithms

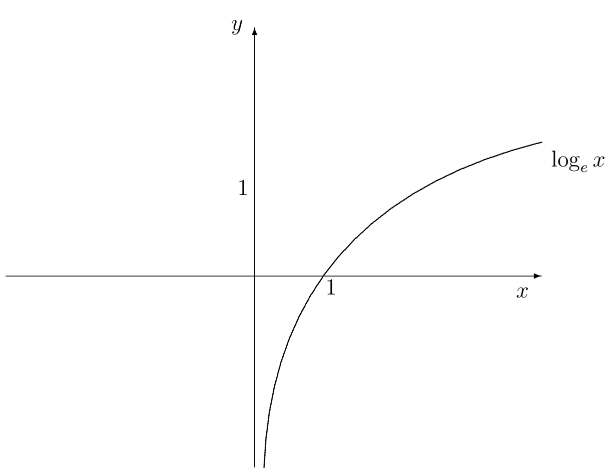
Section 1 Introduction

We have met exponentials and logarithms before in previous modules. Before working through this worksheet you will want to refresh your knowledge of logarithms and exponentials by working through worksheet 2.7 from module 2. In this worksheet we will consider the special relationship logarithms and exponentials have with each other.

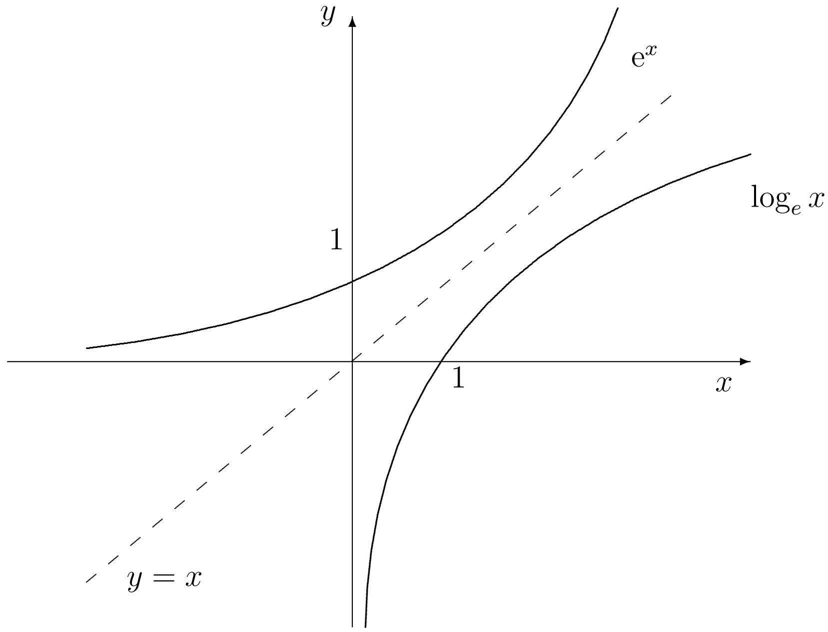
The exponential function e^x looks like



and is a function which grows (or decays) rapidly. The log function $\log_e x$ looks like



We define the log function as the inverse of the exponential, so logs and exponentials "undo" each other. When we sketch their graphs we can easily see the property of inverse functions that they are symmetrical about $y = x$



More formally $f(x) = \mathrm{e}^x$ and $f^{-1}(x) = \log_e x$ whence

$$(f \circ f^{-1})(x) = \mathrm{e}^{\log x} = x \quad \text{and} \quad (f^{-1} \circ f)(x) = log_e(\mathrm{e}^x) = x.$$

Exercises:

1. Use the following exercises to revise your knowledge of exponentials and logarithms.

   (a) Simplify $(4x^3y^{-1})^{\frac{1}{2}} \times \left(\frac{9x^4}{y^{\frac{2}{7}}}\right)^{\frac{5}{2}}$

   (b) Simplify $3^{2x} \times 27^x$

   (c) Solve $8 \times \left(\frac{1}{16}\right)^x = 4^{x+1} \times \sqrt{32}$

   (d) Find the exact value of $\log_5\left(\frac{1}{\sqrt{125}}\right)$

   (e) Simplify $\log x^3 - \log y + 2\log\left(\frac{y}{x}\right)$

   (f) Solve $\log 4x^4 - 2\log 2x = \log(x+2)$

2. What is the domain and range of $y = \log_5 x$. Write down its inverse function and the corresponding domain and range.

3. Consider the exponential function $f(x) = b^x$ (where $b > 0$). For the following values of b sketch the corresponding exponential functions on the same graph.

   (a) $b = 2$

   (b) $b = \frac{1}{2}$

   (c) $b = 10$

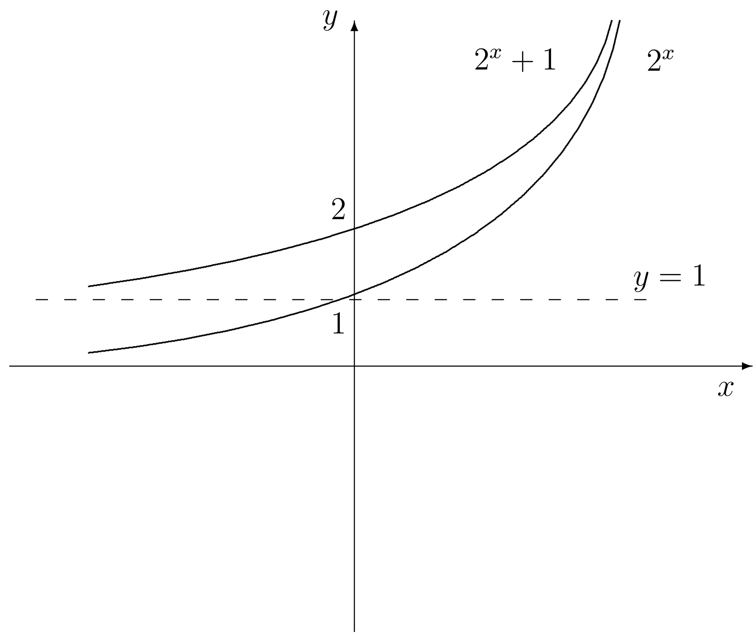
4. For each of the exponentials $f(x)$ you drew in question 3, find its inverse $f^{-1}(x)$ and on a new graph sketch both the function and the inverse you've found.

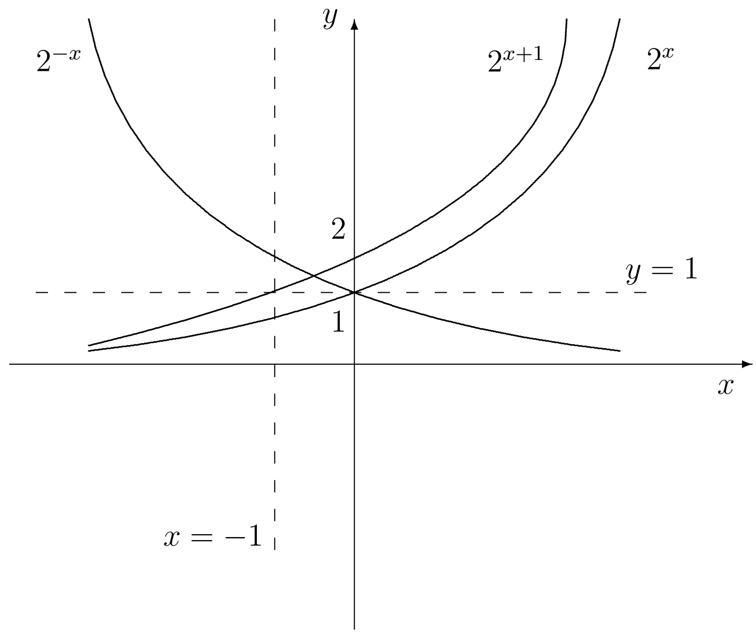
Section 2 Modifying Exponential and Logarithmic Functions

In the same way that we modified functions in Chapter 4.9, we can also easily sketch graphs of modified exponential and logarithmic functions.

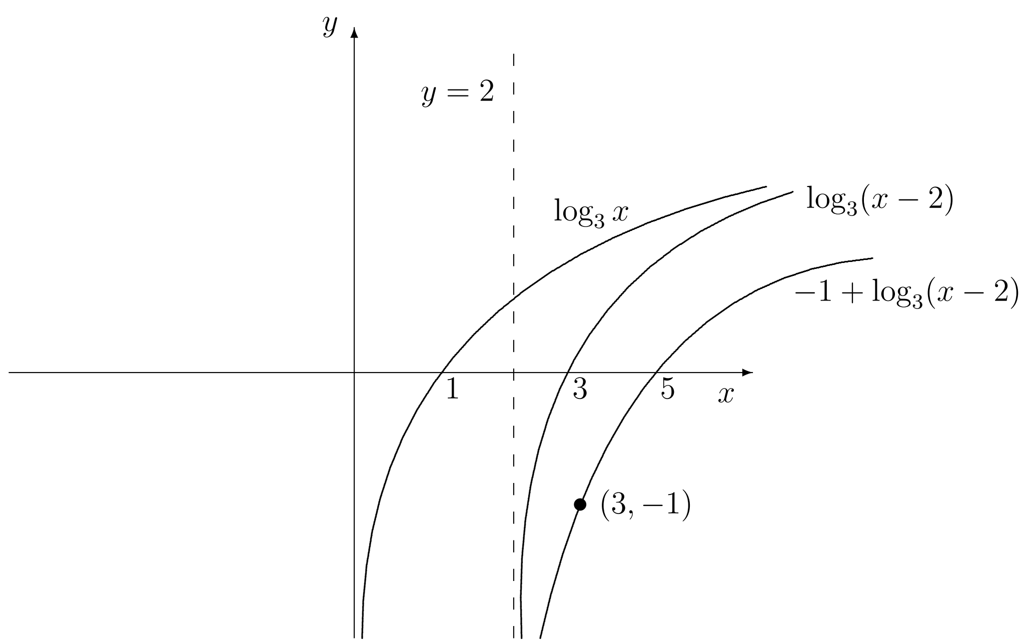
Example 1 : Sketch the graph of $y = 2^x$ and the modifications $y = 2^x + 1, y = 2^{x+1}$ and $y = 2^{-x}$.

The function $y = 2^x$ is essentially the same as $y = e^x$, just not quite as steep, since $e \approx 2.71828$ Notice that $y = 2^x + 1$ is a vertical sift up by 1 unit and that $y = 2^{x+1}$ is a horizontal shift to the left by 1 unit. For $y = 2^{-x}$ what happens for negative x values in this graph is what happened for the positive x values in the original graph. So this is just a reflection about the y-axis.





Example 2 : Sketch the graph $y = \log_3 x$ and $y = -1 + \log_3(x-2)$ on the same axis.
$\log_3 x$ has the same shape as $\log_e x$, increasing at a slightly slower rate. To modify the function we shift it to the right by 2, and subtract 1.



Exercises:

1. Sketch the following exponential functions.
   (a) $y = 3^x$
   (b) $y = 3^{x+1}$
   (c) $y = 1 + 3^x$
   (d) $y = 3^{-x}$
   (e) $y = -3^x$
   (f) $y = \frac{3^x}{3}$
2. Sketch the following functions and state the domain of each.
   (a) $y = \log(x + 3)$
   (b) $y = 1 + \log(x + 3)$
   (c) $y = \log|x + 3|$
   (d) $y = |\log(x + 3)|$
3. Find the domain of the following functions.
   (a) $y = \log(x - 5)$
   (b) $y = \log\sqrt{x - 5}$
   (c) $y = \log|x - 5|$
   (d) $y = \log\sqrt{x^2 - 4}$

Section 3 Exponentials and Logarithms as Inverse Functions

Since the exponential and logarithm functions are inverses we can use the same methods in the previous worksheet to find inverses.

Example 1 : Find the inverse of $f(x) = \log(x + 1)$.
Let $y = \log(x + 1)$. Write $x = \log(y + 1)$ and rearrange

$$\begin{aligned} e^x &= y + 1 \\ y &= e^x - 1, \end{aligned}$$

so

$$f^{-1}(x) = e^x - 1.$$

Example 2 : Find the inverse of $f(x) = \log(x-3)$.

Let $y = \log(x-3)$. Write $x = \log(y-3)$ and rearrange

$$\begin{aligned} e^x &= y - 3 \\ y &= e^x + 3, \end{aligned}$$

so

$$f^{-1}(x) = e^x + 3.$$

Example 3 : Find the inverse of $g(x) = e^{x+1}$.

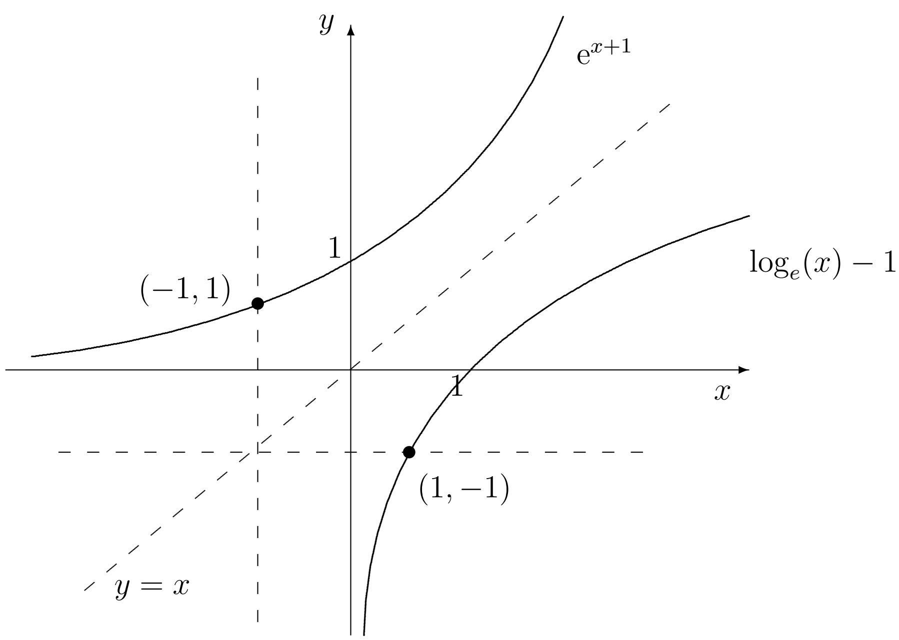
Let $y = e^{x+1}$. Write $x = e^{y+1}$ and rearrange

$$\begin{aligned} \log x &= y + 1 \\ y &= \log(x) - 1, \end{aligned}$$

so

$$g^{-1}(x) = \log(x) - 1.$$

- Note: $g : \mathbb{R} \longrightarrow [0, \infty)$ and $g^{-1} : [0, \infty) \longrightarrow \mathbb{R}$.
- We can see the graphical property of inverse functions by sketching both g and g^{-1} on the same diagram:



Example 4 : Find the inverse of $g(x) = \mathrm{e}^{2x-3} + 2$.

Let $y = \mathrm{e}^{2x-3} + 2$. Write $x = \mathrm{e}^{2y-3} + 2$ and rearrange

$$\begin{aligned} 2y - 3 &= \log(x-2) \\ 2y &= \log(x-2) + 3 \\ y &= \frac{\log(x-2)+3}{2}, \end{aligned}$$

so

$$g^{-1}(x) = \frac{\log(x-2)+3}{2}.$$

Exercises:

1. Consider $f(x) = e^{x-2}$.
   (a) Find the inverse of $f(x)$.
   (b) Verify that this is indeed the inverse of $f(x)$ by considering $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$.
   (c) Sketch $f(x)$ and $f^{-1}(x)$ on the same graph. What do you notice?
2. Find the inverse of the following.
   (a) $y = e^{\frac{1}{x}}$
   (b) $y = \log(3 - x)$
   (c) $y = 2 - \log(x + 1)$
3. Find the inverses of the following functions (with given domain) and specify the domains of the corresponding inverse functions.
   (a) $f(x) = e^{2x}$, domain $[1, \infty)$
   (b) $f(x) = \log(2x + 4)$, domain $(-2, \infty)$
4. Explain why $e^{\log(-1)} \neq -1$.

Exercises for Worksheet 4.9

1. Find the domains and ranges of:
   (a) $\log(2x+3)$
   (b) $\log(\mathrm{e}^x - 1)$
   (c) e^{2x-1}
   (d) $\log(\sqrt{2x-1}+3)$
2. Sketch the following functions.
   (a) $y = \mathrm{e}^{x+4} - 5$
   (b) $y = \log(x-3)$ and $y = \log x - 3$
   (c) $y = \log_e(-x)$
   (d) $y = \mathrm{e}^{-x+1}$
   (e) $y = \frac{1}{e^x - 2}$
3. Sketch the following functions.
   (a) $y = |2\log(x-3)|$
   (b) $y = |3\mathrm{e}^{x+5} - 7|$
   (c) $y = \left|\frac{1}{e^x-2}\right|$
4. Find the inverses of the following functions
   (a) $y = \mathrm{e}^{x-2}$
   (b) $y = \mathrm{e}^{x+4} - 5$
   (c) $y = \log_e x - 2$
   (d) $y = \log_e(-x)$
   (e) $y = 3 + \log_2(x+4)$
   (f) $y = 2^{-x+1}$
5. Let $g(x) = \log x$ and $h(x) = x - 2$.
   (a) Find $k(x) = (g \circ h)(x)$.
   (b) Find the domain of $k(x)$.
   (c) Find the range of $k(x)$.
   (d) Find $k^{-1}(y)$.
   (e) Sketch $k(x)$ and $k^{-1}(y)$ on the same plane.
6. (Harder) Find the inverse of $f(x) = \log|x+3|$.

Answers for Worksheet 4.9

Section 1

1. (a) $486x^{\frac{23}{2}}y^{-\frac{17}{14}}$
   (b) 3^{5x}
   (c) $-1/4$
   (d) $-3/2$
   (e) $\log x + \log y$
   (f) 2

2. The domain of $y = \log_5 x$ is $(0, \infty)$ and its range is $\mathbb{R}$. Its inverse is $y = 5^x$ with domain $\mathbb{R}$ and range $(0, \infty)$.

Section 2

2. (a) $(-3, \infty)$
   (b) $(-3, \infty)$
   (c) $(-3, \infty)$
   (d) $\{x \in \mathbb{R} \mid x \neq 3\}$

3. (a) $(5, \infty)$
   (b) $(5, \infty)$
   (c) $\{x \in \mathbb{R} \mid x \neq 5\}$
   (d) $(-\infty, -2) \cup (2, \infty)$

Section 3

1. (a) $f^{-1}(x) = \log x + 2$

2. (a) $\dfrac{1}{\log x}$
   (b) $3 - e^x$
   (c) $e^{2-x} - 1$

3. (a) $\frac{1}{2}\log x$; $[e^2, \infty)$
   (b) $\dfrac{e^x}{2} - 2$; $\mathbb{R}$

4. One reason is that $\log(-1)$ is not defined. Another reason is that the function e^x is always positive valued.

Exercises 4.9

1. The domain and range are, respectively,

   (a) $(-\frac{3}{2}, \infty)$ and $\mathbb{R}$

   (b) $(0, \infty)$ and $\mathbb{R}$

   (c) $\mathbb{R}$ and $(0, \infty)$

   (d) $[\frac{1}{2}, \infty)$ and $[\log 3, \infty)$

4. (a) $\log x + 2$

   (b) $\log(x + 5) - 4$

   (c) e^{x+2}

   (d) $-e^x$

   (e) $2^{x-3} - 4$

   (f) $1 - \log_2 x$

5. (a) $k(x) = \log(x - 2)$

   (b) $(2, \infty)$

   (c) $\mathbb{R}$

   (d) $k^{-1}(x) = e^x + 2$

6. There are two possibilities.

   (a) If we restrict the domain of f to $(-3, \infty)$ then its inverse will be $f^{-1}(x) = e^x - 3$.

   (b) If we restrict the domain of f to $(-\infty, -3)$ then its inverse will be $f^{-1}(x) = -e^x - 3$.