Worksheet 4.8 Composite and Inverse Functions

Section 1 Composition

We'll begin by defining the composition function $f \circ g(x) = f(g(x))$, which is read as "f of g of x". Another helpful way to think about these is to call them "a function (f) of a function (g)". To calculate this function for a given x, first evaluate $g(x)$, which will give us a number, and then take this number and apply the function f to it giving $f(g(x))$. Note: For $f \circ g(x)$ to exist x must be in the domain of g and $g(x)$ must be in the domain of f. This is expanded on in Section 3 of Chapter 3.1.

Example 1 : Find the composition of the two functions $f(x) = x^2 - 2x + 3$ and $g(x) = 2x - 1$.

$$(f \circ g)(x) = (2x-1)^2 - 2(2x-1) + 3 = 4x^2 - 8x + 6$$

$$(g \circ f)(x) = 2(x^2 - 2x + 3) - 1 = 2x^2 - 4x + 5.$$

Notice that $f \circ g \neq g \circ f$.

Example 2 : Consider the functions $p(x) = \sqrt{x-1}$ and $q(x) = x^2$. Find $(q \circ p)(x)$.

Firstly notice $p(x)$ has domain $[1, \infty)$ and range $[0, \infty)$ and $q(x)$ has domain $\mathbb{R}$ and range $[0, \infty)$.

Look at $(q \circ p)(x) = q(p(x)) = (\sqrt{x-1})^2 = x - 1$. When finding the domain of $(q \circ p)(x)$ we need to consider what is happening in this composite. Here the function $p(x)$ is applied first, followed by $q(x)$.

So we start by inputting the domain $[1, \infty)$ of $p(x)$ such that its range $[0, \infty)$ becomes the input of $q(x)$. Then $q(x)$ squares these values to give the overall output of the composite. Stringing these observations together tells us that the domain of $(q \circ p)(x)$ is $[1, \infty)$ and its corresponding range is $[0, \infty)$.

Notice that we are incorrectly tempted to look at $(q \circ p)(x) = x - 1$ and claim its domain is $\mathbb{R}$ and its range is $\mathbb{R}$. However as this is a composite of two functions its input and output values depend on the domain and range of the two individual functions $q(x)$ and $p(x)$.

Sometimes it is necessary to restrict the domain of g, so that $f \circ g$ will exist.

Exercises:

1. Consider $f(x) = 4 - x^2$, $g(x) = \sqrt{x+3}$, $h(x) = \frac{1}{2x}$. Evaluate the following.
   (a) $(f \circ g)(1)$
   (b) $(g \circ h)(1)$
   (c) $(f \circ g)(x)$
   (d) $(g \circ h)(x)$
   (e) $(h \circ g)(x)$
   (f) $(f \circ g)(x^2)$
   (g) $(f \circ g \circ h)(x)$
2. Using the functions given in the previous exercise, explain why $(f \circ g)(-4)$ does not exist.
3. Let $s(x) = \sqrt{x}$ and $t(x) = x^2 + 2x + 1$. Evaluate $(s \circ t)(x)$ and state its domain and range.
4. $f(x) = \sqrt{\frac{1}{x^2+2}}$. Write $f(x)$ as the composition of two or more functions.

Section 2 Inverse Functions

Let us introduce the concept of inverse functions by looking at some examples.

Example 1 : $f(x) = x + 2, \quad g(x) = x - 2$
$f(x)$ adds 2 to everything we put into it.
$g(x)$ subtracts 2 from everything we put into it.
What happens when we take $f \circ g$?
$f(g(x)) = f(x-2) = x - 2 + 2 = x$. $(f \circ g)(x)$ takes the input x, first subtracts 2 then adds 2 so we are back to what we started with.
Also $(g \circ f)(x) = g(f(x)) = g(x+2) = x + 2 - 2 = x$. So $f(x)$ and $g(x)$ "undo" each other.

Example 2 : $h(x) = 3x, \quad k(x) = \frac{x}{3}$
Here $h(x)$ multiplies what we put in by 3 and $k(x)$ divides what we put in by 3.
$(h \circ k)(x) = h\left(\frac{x}{3}\right) = 3\frac{x}{3} = x$ and $(k \circ h)(x) = k(3x) = \frac{(3x)}{3} = x$.
So $(h \circ k)(x)$ takes the input x, divides by 3, then multiplies by 3, thus returning back to x. $(k \circ h)(x)$ takes the input x, multiplies by 3, then divides by 3, thus returning back to x. So $h(x)$ and $k(x)$ "undo" each other.

Notice in each of examples 1 and 2 the operations performed are "opposites" so the functions "undo" each other. Similarly we have example 3.

Example 3 : $p(x) = x^3, \quad q(x) = x^{1/3}$
$p(x)$ cubes what we put in and $q(x)$ takes the cube root of what we put in. So $p(x)$ and $q(x)$ will undo each other and $(p \circ q)(x) = x = (q \circ p)(x)$.

We can see that certain pairs of functions have the important property that they undo each other. We call functions that undo each other *inverses*. So $p(x)$ and $q(x)$ are inverses since $(p \circ q)(x) = x$ and $(q \circ p)(x) = x$.

To represent this special relationship they have with each other we rename them. For $p(x) = x^3$ we say

$$p^{-1}(x) = x^{\frac{1}{3}}.$$

Here we use a superscript of -1 to represent that we are talking about the inverse of p. [NOTE: $p^{-1}(x) \neq \frac{1}{p(x)}$] So we say that $(p \circ p^{-1})(x) = (p^{-1} \circ p)(x) = x$.

Definition 1 : The inverse function for f is the unique function f^{-1} that satisfies $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

Looking back, from example 1 we have $f(x) = x + 2$, $f^{-1}(x) = x - 2$ and $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. While from example 2 we have example 2) $h(x) = 3x$, $h^{-1}(x) = \frac{x}{3}$ and $(h \circ h^{-1})(x) = (h^{-1} \circ h)(x) = x$.

Exercises:

1. Show the following function pairs are inverses:
   (a) $f(x) = x^2$ and $g(x) = \sqrt{x}$
   (b) $f(x) = \frac{1}{x^2-1}$ and $g(x) = \sqrt{\frac{1}{x} + 1}$
   (c) $f(x) = 2x - 2$ and $g(x) = \frac{x}{2} + 1$
2. Which pairs of the following functions are inverses?
   (a) $y = x^2 - 2$
   (b) $y = x - 7$
   (c) $y = x^2 - 4x + 4$

(d) $y = \sqrt{x} + 2$

(e) $y = \sqrt{x + 2}$

(f) $y = x + 7$

Section 3 Finding Inverses

There is a particular technique we can use to find inverses of functions. Let's look at an example to see how this works.

Example 1 :

$$f(x) = \frac{2x - 3}{7}.$$

To find the inverse of $f(x)$, let's rewrite it so that we call the output values y, i.e. $y = \frac{2x-3}{7}$. Since the inverse function will undo the original, we expect the outputs of the inverse to bring us back to the inputs of the original, and vice versa. So for our inverse function we expect

$$x = \frac{2y - 3}{7}$$

i.e. we swap the x and y values which represent the inputs and outputs. To find the inverse function we now make y the subject.

$$\begin{aligned} x &= \frac{2y - 3}{7} \\ 7x &= 2y - 3 \\ 7x + 3 &= 2y \\ y &= \frac{7x + 3}{2}, \end{aligned}$$

so our inverse function is $y = \frac{7x+3}{2}$. We write $f^{-1}(x) = \frac{7x+3}{2}$.

Notice we have a simple technique to find inverses - we swap x and y in the original function and then rearrange to make y the subject.

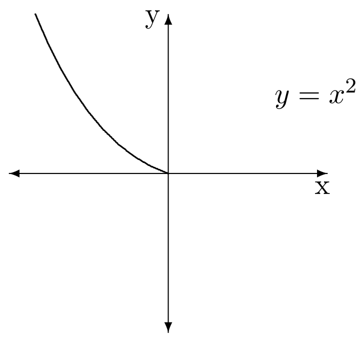
Example 2 : $f(x) = 3x - 11$
Let $y = 3x - 11$. To find the inverse we write $x = 3y - 11$ and rearrange $x + 11 = 3y$ giving $y = \frac{x+11}{3}$. So

$$f^{-1}(x) = \frac{x+11}{3}.$$

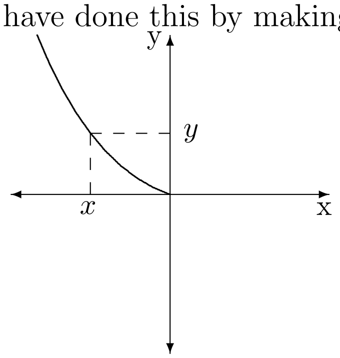
Suppose $f : A \to B$ where A is the domain and B is the range. Then $f^{-1} : B \to A$. In fact we have to take particular note of the domain and range when finding inverse functions especially in certain situations.

Consider $f(x) = x^2$ where $f : \mathbb{R} \to [0, \infty)$. To find the inverse we write $y = x^2$ then swap $x = y^2$ then rearrange $y^2 = x$ to get $y = \pm\sqrt{x}$. Here $f^{-1}(x) = +\sqrt{x}$ and $f^{-1}(x) = -\sqrt{x}$. We cannot have 2 different inverse functions for $f(x) = x^2$. If $f^{-1}(x)$ is both $\pm\sqrt{x}$ then it is not a function at all. So in this situation we need to make sure that $f(x)$ is defined correctly in the first place so that the inverse can be taken. Since we don't want two different inverses, we need to restrict the domain of $f(x)$.

Let's take $f : (-\infty, 0] \to [0, \infty), f(x) = x^2$



Therefore $f^{-1} : [0, \infty) \to (-\infty, 0]$. By restricting the domain we have removed of the problem of getting more than 1 inverse. We have done this by making sure that for each y-value there is only one corresponding x-value.

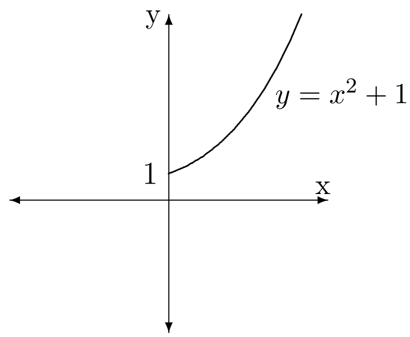


Now by looking at the range of f^{-1} (the negative reals) we can therefore see that we need $f^{-1}(x) = -\sqrt{x}$ here.

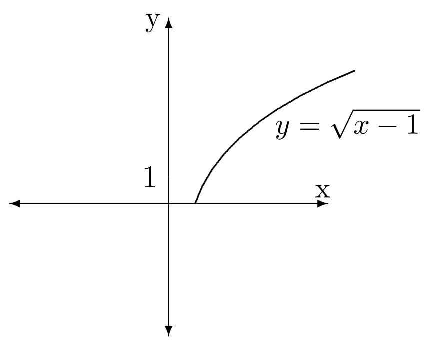
There are times that we need to restrict the domain and codomain of a function so that is is possible to find its inverse. In the previous example we have done this by making sure f is <u>one-to-one</u> i.e. made sure that for each output there is only one input corresponding to it. We also need our function to be <u>onto</u> i.e. the function gives outputs across the whole domain, meaning that the codomain = range. (We didn't have to worry about this in the previous example as the codomain = range was given). By restricting our function so that it is both 1-1 and onto we will be able to find an inverse.

Example 3 : Find the inverse of the function $f(x) = x^2 + 1$ with domain $[0, \infty)$ (so that the range of f is $[1, \infty)$).

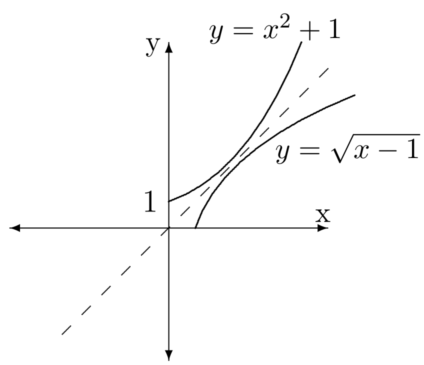
From a quick sketch we see that f is 1-1 and onto.



We need $f : [0, \infty) \to [1, \infty)$ so $f^{-1} : [1, \infty) \to [0, \infty)$. We find the inverse as before, start by swapping variables to $x = y^2 + 1$ and then rearrange to get $y = \pm\sqrt{x-1}$. By looking at the range of f^{-1}, the positive reals, we see that we need to take $y = +\sqrt{x-1}$ as the inverse i.e. $f^{-1}(x) = \sqrt{x-1}$.



We can observe that a special property of inverse functions is that they are symmetrical about the line $y = x$



Exercises:

1. Find the inverses of these functions and sketch each function and its inverse on the same graph.

   (a) $y = 3x$

   (b) $y = x - 7$

   (c) $y = \frac{x}{2} + 17$

   (d) $y = \frac{1-5x}{4}$

   (e) $y = \frac{3}{x+8}$

   (f) $y = x^3$

2. Find the inverses of these functions and state their domain and range.

   (a) $y = \sqrt{x} + 2$

   (b) $y = \sqrt{x + 2}$

3. By suitably restricting the domain, find the inverse of $y = x^2 - 2$ and state its domain and range.

Exercises for Worksheet 4.8

1. For the following pairs of functions f and g find the composition functions $f \circ g$ and $g \circ f$. Also find the domains of the two composition functions.

   (a) $f = 2x - 5$ and $g = x^2 - 3x$

   (b) $f = \sqrt{3x-1}$ and $g = x^2$

   (c) $f = \sqrt{3x-1}$ and $g = \frac{1}{x}$

   (d) $f = \sqrt{3x-1}$ and $g = \frac{x^2+1}{x}$

   (e) $f = x^2 - 3$ and $g = |x|$

2. Write each of the following functions as the composition of two or more simpler functions.

   (a) $\sqrt{x^3 - 1}$

   (b) $(3x-4)^3$

   (c) $\frac{1}{x^2-1}$

   (d) x

3. Find the inverses of the following functions.

   (a) $y = 3x + 2$

   (b) $y = \frac{1}{4-x}$

   (c) $y = \frac{x+2}{x+5}$

   (d) $y = x^3 + 1$

4. Find the inverses of the following functions and specify the domain of these inverses.

   (a) $f(x) = x^2 - 2x + 3$, domain $[1, \infty)$

   (b) $h(x) = |x|$, domain $(-\infty, 0)$

   (c) $g(x) = 9 - x^2$, domain $\mathbb{R}^+$

Answers for Worksheet 4.8

Section 1

1. (a) 0
   (b) $\sqrt{\frac{7}{2}}$
   (c) $1-x$
   (d) $\sqrt{\frac{1}{2x}+3}$
   (e) $\frac{1}{2\sqrt{x+3}}$
   (f) $1-x^2$
   (g) $1-\frac{1}{2x}$

2. The number -4 is not in the domain of g.

3. We have $(s \circ t)(x) = \sqrt{x^2+2x+1}$. Its domain is $\mathbb{R}$ and its range is $[0, \infty)$.

4. We have $f(x) = (g \circ h)(x)$ where $g(x) = \sqrt{x}$ and $h(x) = \frac{1}{x^2+2}$. Another possibility is $g(x) = \sqrt{\frac{1}{x+2}}$ and $h(x) = x^2$.

Section 2

2. (a) and (e); (b) and (f); (c) and (d).

Section 3

1. (a) $y = \frac{x}{3}$
   (b) $y = x + 7$
   (c) $y = 2(x-17)$
   (d) $y = -\frac{4}{5}x + \frac{1}{5}$
   (e) $y = \frac{3}{x} - 8$
   (f) $y = x^{\frac{1}{3}}$

2. (a) $y = (x-2)^2$ with domain $[2, \infty)$ and range $[0, \infty)$.
   (b) $y = x^2 - 2$ with domain $[0, \infty)$ and range $[-2, \infty)$.

3. There are two possibilities.
   (a) Restrict the domain of $y = x^2 - 2$ to $[0, \infty)$. The inverse is $y = \sqrt{x+2}$ with domain $[-2, \infty)$ and range $[0, \infty)$.
   (b) Restrict the domain of $y = x^2 - 2$ to $(-\infty, 0]$. The inverse is $y = -\sqrt{x+2}$ with domain $[-2, \infty)$ and range $(-\infty, 0]$.

Exercises 4.8

1. The compositions and their domains are:

	Function	Equation	Domain
(a)	$(f \circ g)$	$2x^2 - 6x - 5$	$\mathbb{R}$
	$(g \circ f)$	$4x^2 - 26x + 40$	$\mathbb{R}$
(b)	$(f \circ g)$	$\sqrt{2x^2 - 1}$	$(-\infty, -\frac{1}{\sqrt{3}}] \cup [\frac{1}{\sqrt{3}}, \infty)$
	$(g \circ f)$	$3x - 1$	$[\frac{1}{3}, \infty)$
(c)	$(f \circ g)$	$\sqrt{\frac{3}{x} - 1}$	$(-\infty, 0) \cup (0, 3)$
	$(g \circ f)$	$\frac{1}{\sqrt{3x - 1}}$	$(\frac{1}{3}, \infty)$
(d)	$(f \circ g)$	$\sqrt{\frac{3x^2 - x + 3}{x}}$	$(0, \infty)$
	$(g \circ f)$	$\frac{3x}{\sqrt{3x - 1}}$	$(\frac{1}{3}, \infty)$
(e)	$(f \circ g)$	$x^2 - 3$	$\mathbb{R}$
	$(g \circ f)$	$\lvert x^2 - 3\rvert$	$\mathbb{R}$

2. Some possibilities are:

   (a) $(f \circ g)(x) = \sqrt{x^3 - 1}$ where $f(x) = \sqrt{x}$ and $g(x) = x^3 - 1$,

   (b) $(f \circ g)(x) = (3x - 4)^3$ where $f(x) = x^3$ and $g(x) = 3x - 4$,

   (c) $(f \circ g)(x) = \dfrac{1}{x^2 - 1}$ where $f(x) = \dfrac{1}{x}$ and $g(x) = x^2 - 1$,

   (d) $(f \circ g)(x) = x$ where $f(x) = x^2$ and $g(x) = \sqrt{x}$.

3. (a) $\dfrac{x}{3} - \dfrac{2}{3}$ (b) $-\dfrac{1}{x} + 4$ (c) $\dfrac{3}{1 - x} - 5$ (d) $(x - 1)^{1/3}$

4. (a) $f^{-1}(x) = \sqrt{x - 2} + 1$ with domain $[2, \infty)$.

   (b) $h^{-1}(x) = -x$ with domain $(0, \infty)$.

   (c) $g^{-1}(x) = \sqrt{9 - x}$ with domain $(-\infty, 9]$.