Worksheet 4.7 Modifying Functions - Families of Graphs

Section 1 Domain, range and functions

We first met functions in Sections 3.1 and 3.2. We will now look at functions in more depth and discuss their domain and range more formally.

Defining Functions

A function f is specified by a rule and two sets. These sets are the *domain*, previously discussed in worksheet 3.2, which we'll call A, and the *codomain* which we'll call B. The domain of a function contains all the values that we can input into the function. The codomain is a set containing all possible values the function could achieve. The codomain is usually given.

We can think of our function as a mapping from the domain to the codomain and we usually write this as

$$f : A \to B$$

where

$$A = \text{domain}$$
$$B = \text{codomain.}$$

Example 1 : Consider $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$.

We say that f takes real numbers (from the domain), applies the rule and maps them to other real numbers in the codomain. Here the rule says that we square what we put into our function.

So for example $f(2) = 4$. That is, f maps 2 from the domain to 4 in the codomain. Another example is $f(-1) = 1$. In this case f maps -1 in the domain to 1 in the codomain.

Example 2 : Consider $f : [-1, 3) \to [1, 10)$, where $f(x) = x^2 + 1$.

The above function f has domain $[-1, 3)$ and codomain $[1, 10)$.
The rule of this function is that we take some element x of the domain and then evaluate $x^2 + 1$ to find $f(x)$. For example $f(\frac{1}{2}) = 1\frac{1}{4}$.

The last set we will need to define is the *range* of a function. The range, also talked about in worksheet 3.2, is a subset of the codomain, and contains all of the values that f actually attains. The range is always contained in the codomain, but the codomain might not necessarily be in the range (it can contain more things). In Example 2 the range of f is the same as the codomain, $[1, 10)$.

Example 3 : Consider again $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$.

Since we square all our inputs, we will never get a negative number as an output. In other words, $f(x)$ will only attain 0 and positive values. So the range of $f(x)$ is $[0, \infty)$. Notice this is a subset of the given codomain $\mathbb{R}$.

Example 4 : Consider $f : [0, 3) \to \mathbb{R}$, where $f(x) = x^3 - 7$. Here the domain is $[0, 3)$, the range is $[-7, 20)$ and the codomain is $\mathbb{R}$ (given). Note that the range is a subset of the codomain.

All functions have only one value of $f(x)$ for each value of x.

Example 5 : Find the domain of $f(x) = \frac{1}{x-5}$.
We can never divide by 0, thus for x to be in the domain it must satisfy $x - 5 \neq 0$. So the domain of the function is $x \neq 5$.

Set Notation

Another way of writing the domain from Example 5 is to use set notation. We say

$$\text{Domain} = \{x \in \mathbb{R} : x \neq 5\},$$

which we read as 'the set of all real numbers x, such that x is not equal to 5'. We use the brackets $\{\dots\}$ to respresent our set and the symbol $\in$ to mean 'belongs to'. The colon ' : ' may be read as either 'such that' or 'with the property that'.

Example 6 : Find the domain and range of $f(x) = \sqrt{x+3}$.
We can only take the square root of positive numbers or 0, so to be in the domain x must satisfy $x + 3 \geq 0$, which gives domain $[-3, \infty)$. The square root function always gives a positive or 0 result, so $\sqrt{x+3} \geq 0$ so the range of the function is $[0, \infty)$.

We can write this using set notation as

$$\text{Domain} = \{x \in \mathbb{R} : x \geq -3\}$$
$$\text{Range} = \{y \in \mathbb{R} : y \geq 0\}.$$

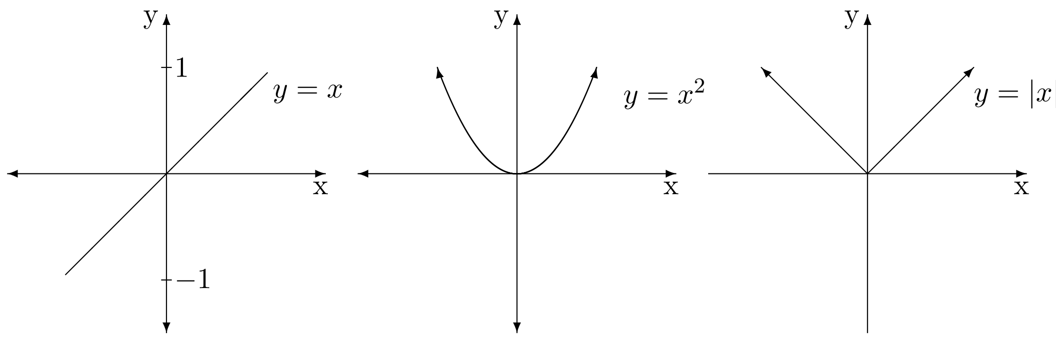
Exercises:

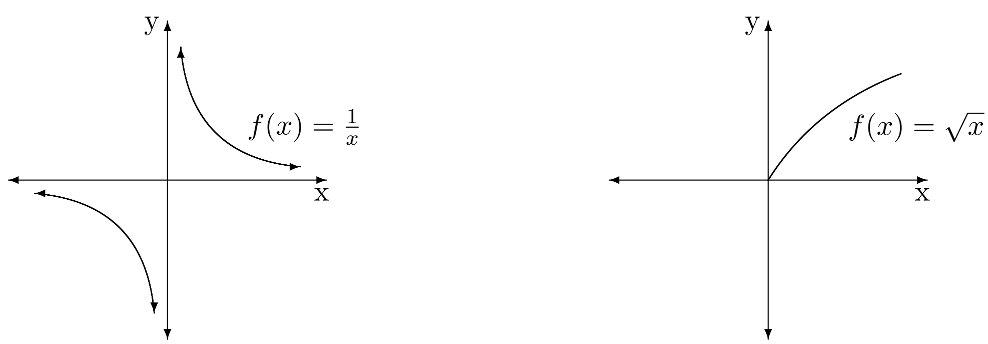
1. State the domain and codomain and find the range of the following functions
   (a) $f : (0,5) \to \mathbb{R}, f(x) = x^2 - 2.$
   (b) $g : [-1,7] \to \mathbb{R}, g(x) = x^2 - 2x + 1.$
2. Find a suitable domain and the related codomain for $f(x) = \sqrt{x-1}$.
3. Find the domain of the functions below and express it in set notation.
   (a) $f(x) = \sqrt{x}$
   (b) $f(x) = \frac{1}{\sqrt{x}}$
   (c) $f(x) = \frac{1}{3x-2}$
   (d) $f(x) = x^3 - 1$
   (e) $f(x) = \sqrt{x-7}$
   (f) $f(x) = \sqrt{x^2 - 16}$
4. Find the domain and range of
   (a) $f(x) = x^2 + 1$
   (b) $f(x) = \sqrt{x+9}$
   (c) $f(x) = 4x - 2$

Section 2 Modifying Functions – Translations

You may have noticed that it is often easier to find the range of a function by looking at its graph. In fact the graph of a function can give us the big picture on how the function behaves. The next two sections look at modifying known functions to quickly and easily sketch other graphs.

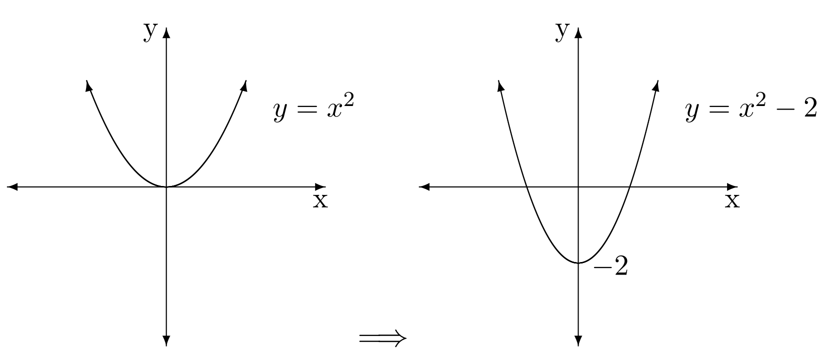
We know how to draw the standard graphs of some basic functions, for example





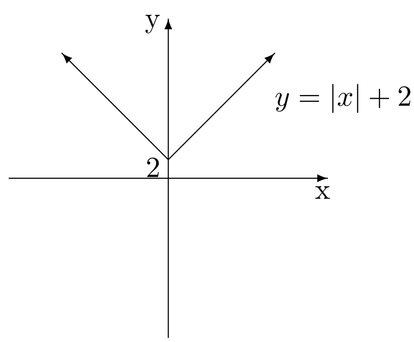
This worksheet will show you how to easily and quickly draw modified versions of these graphs.

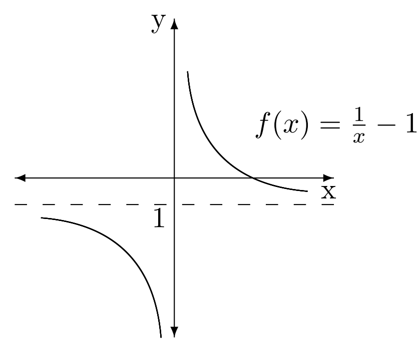
The first kind of modification is one that occurs on the y-coordinate. For example we might want to sketch $y = x^2 - 2$. This graph is taken by picking some value of x, squaring it and then subtracting 2. This is the same for every value of x, so this is just the graph $y = x^2$ shifted down by 2 units.



Definition 1 : The modification $y = f(x) + a$ is drawn by shifting the graph of $y = f(x)$ up by a units. Similarly the modification $y = f(x) - b$ is drawn by shifting the graph down b units.

Example 1 : Sketch the graphs $y = |x| + 2$ and $y = \frac{1}{x} - 1$. The first graph is a shift up of 2 units, and the second graph is a shift down of one unit.





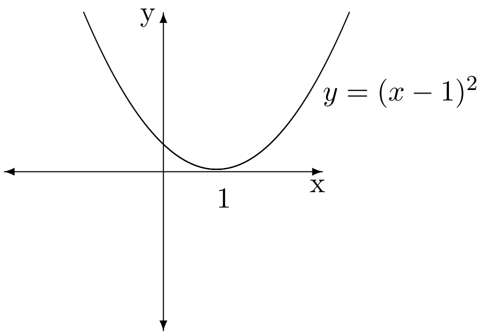
The other kind of modification is one that occurs on the x coordinate. For example suppose we want to sketch

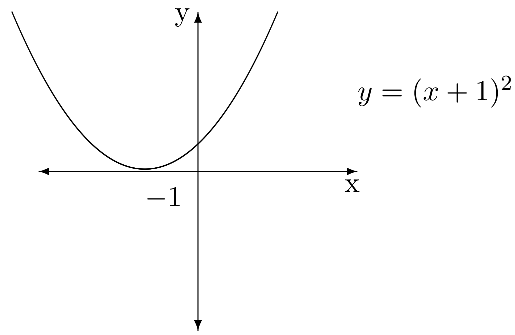
(a) $y = (x-1)^2$

(b) $y = \frac{1}{x+2}$

(c) $y = \sqrt{x-3}$

The key thing to notice here is that the order of operations tells us that in (a) we subtract 1 from x and then square the expression. Similarly in (b) we add 2 to x and then apply the function. Finally in (c) we subtract 3 from x and then take the square root. All of these amount to essentially the same thing, i.e. we are adding or subtracting a number before applying the basic function. To see this, think for a moment about how the graphs are drawn. Let's look at (a) - going from the original function $y = x^2$ to the new function $y = (x-1)^2$. In the original function we take some particular value for x, let's say $x = 4$, and **then** square it to get the y value. So we have a point on the original graph $(4, 16)$. In comparison, looking at our new function and taking the same x value, we subtract 1 from it **before** squaring so the y value is $y = 3^2 = 9$ and the point on our new graph is $(4, 9)$. So the y value is the same as the y value 1 unit to the left in the non-modified graph.

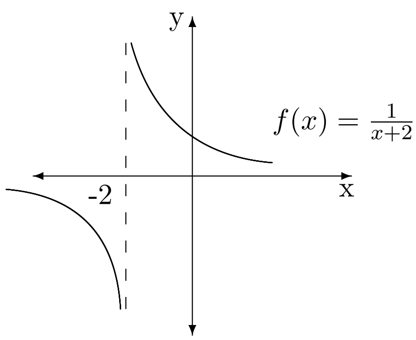


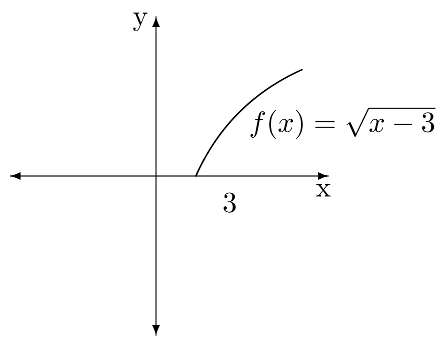


A simple way to state a rule would be to say:

> **Definition 2** : For a modification to a function on the x coordinate of $f(x+a)$ the graph is drawn by shifting $f(x)$ to the left by a units. Similarly the graph $f(x-b)$ is drawn by shifting $f(x)$ to the right by b units.

The other graphs look like:





The modifications we have looked at in this section have either shifted our original function vertically or horizontally. We call this kind of modification a translation.

Exercises:

1. On the same diagram sketch the graphs of $y=x^2$ and $y=x^2+3$.
2. Sketch

   (a) $y=\sqrt{x+5}$
   (b) $y=\sqrt{x}+5$
   (c) $y=|x+20|$
   (d) $y=|x|+20$
   (e) $y=\frac{1}{x-7}$
   (f) $y=3+\frac{1}{x-7}$

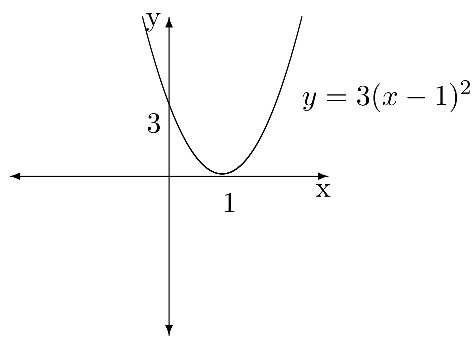
Section 3 Other Modifications

There are three more standard modifications to consider, the first is multiplying the function by a constant. This modification takes the original y values of the functions and changes them by the scalar that we are multiplying by.

Example 1 :

$$y = 3(x-1)^2$$

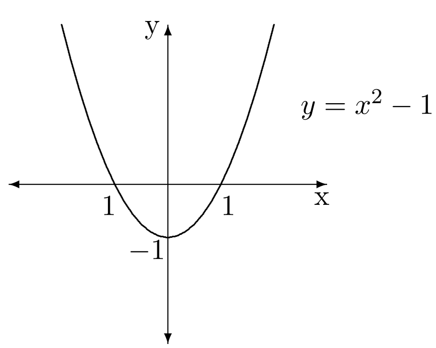
In this example the number 3 is the scalar of multiplication.

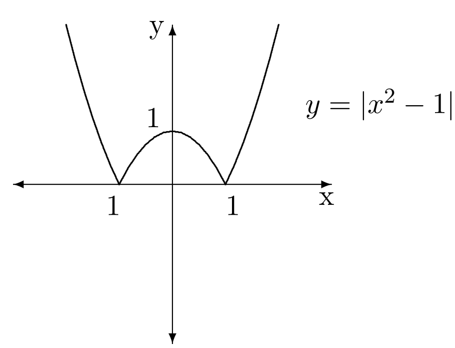


The only difference between this example and the previous drawing of $y = (x-1)^2$ is that this function is steeper

The next modification we'll talk about is taking absolute values. This modification takes any positive y values and leaves them unchanged, and takes any negative values making them positive with the same value. This is the same as putting a mirror along the x axis, and drawing any values below the axis at their mirrored position above the x axis.

Example 2 : Sketch the graph of $y = |x^2 - 1|$.





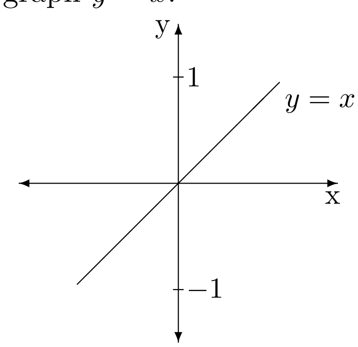
The last modification we will look at is taking reciprocals. The reciprocal of x is $\frac{1}{x}$, we just flip the fractions over (thinking of x as $\frac{x}{1}$). Here are some numbers and their reciprocals.

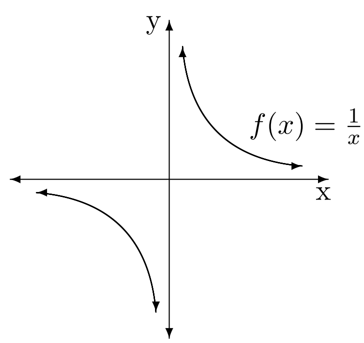
Number (x)	Reciprocal $\frac{1}{x}$	Number (x)	Reciprocal $\frac{1}{x}$
1	1	$\frac{1}{4}$	4
2	$\frac{1}{2}$	$\frac{1}{2}$	2
3	$\frac{1}{3}$	$\frac{1}{100}$	100
10	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{5}{2}$
100	$\frac{1}{100}$	$\frac{2}{55}$	$\frac{55}{2}$

Note that the smaller the number, the larger the reciprocal and the larger the number the smaller the reciprocal.

Note that 0 has no reciprocal, but as x gets closer to 0 its reciprocal gets larger (approaches ∞). If it gets close to zero and is negative then its reciprocal will approach $-\infty$, if it is positive then it approaches ∞. Similarly as x gets large (approaches ∞) the reciprocal approaches 0.

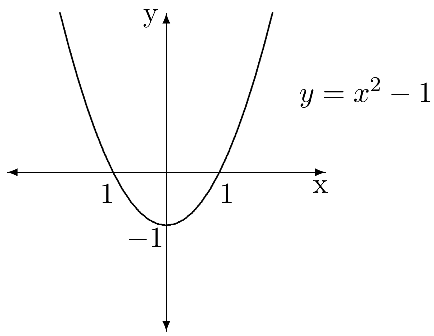
Example 3 : We can see how to draw the reciprocal function $y = \frac{1}{x}$ by using the graph $y = x$.

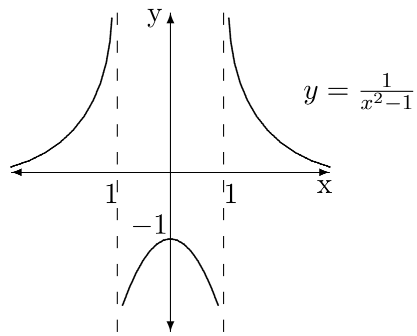




On the graph of $y = \frac{1}{x}$ notice there is a break along the line $x = 0$. This is because x cannot take the value of 0, however as x gets closer to 0, $\frac{1}{x}$ approaches $\pm\infty$ respectively. We call the line $x = 0$ a vertical asymptote.

Example 4 : Sketch the graph of $y = \frac{1}{x^2-1}$.





Exercises:

1. Sketch

   (a) $y = 3(x-2)^2 + 1$

   (b) $y = 7 - \frac{4}{x+3}$

   (c) $y = 2|x-7| + 10$

   (d) $y = \sqrt{3x-2} - 1$

2. Sketch the graph of $y = x^2$. Using this and considering reciprocals sketch the graph of $y = \frac{1}{x^2}$.

3. Let $f(x) = 2 - x^2$. Sketch

   (a) $y = f(x) + 3$

   (b) $y = |f(x)|$

   (c) $y = f(x+3)$

   (d) $y = 1 - f(x)$

4. Sketch $f(x) = \sqrt{x-3} + 2$ and find its domain and range.

Exercises for Worksheet 4.7

1. Sketch the following functions and find their domains and ranges.
   (a) $y = \frac{1}{x-1}$
   (b) $y = |x - 7| - 3$
   (c) $y = \sqrt{3x - 14}$
   (d) $y = x^2 + 6x - 1$ (Hint: complete the square)

2. Sketch the following functions.
   (a) $y = x^2 + 3$
   (b) $y = (x + 3)^2$
   (c) $y = 1 + (x + 3)^2$
   (d) $y = 1 - (x + 3)^2$
   (e) $y = \frac{1}{(x+3)^2}$
   (f) $y = 5(x + 3)^2 - 1$

3. Sketch the following functions.
   (a) $y = 3 + \sqrt{4x - 9}$
   (b) $y = \left|\frac{1}{x^2-4}\right|$
   (c) $y = \frac{x-1}{x+2}$
   (d) $y = \left|\frac{3x-1}{3x-4}\right|$

4. In this question $f(x) = \frac{1}{x-2}$. Sketch the following.
   (a) $y = f(x)$
   (b) $y = f(x + 3)$
   (c) $y = f(x) + 3$
   (d) $y = |f(x) + 3|$
   (e) $y = \frac{1}{f(x)}$

Answers for Worksheet 4.7

Section 1

1. (a) Domain is $(0, 5)$, range is $(-2, 23)$, codomain is $\mathbb{R}$.
   (b) Domain is $[-1, 7]$, range is $[0, 36]$, codomain is $\mathbb{R}$.

2. A suitable domain is $[1, \infty)$ and a suitable codomain is $[0, \infty)$.

3. (a) $\{x \in \mathbb{R} : \ x \geq 0\}$
   (b) $\{x \in \mathbb{R} : \ x > 0\}$
   (c) $\{x \in \mathbb{R} : \ x \neq \frac{2}{3}\}$
   (d) $\mathbb{R}$
   (e) $\{x \in \mathbb{R} : \ x \geq 7\}$
   (f) $\{x \in \mathbb{R} : \ x \leq -4 \quad \text{or} \quad x \geq 4\}$

4. (a) Domain is $\mathbb{R}$. Range is $[1, \infty)$.
   (b) Domain is $[-9, \infty)$. Range is $[0, \infty)$.
   (c) Domain is $\mathbb{R}$. Range is $\mathbb{R}$.