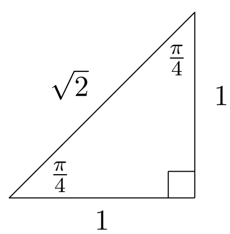
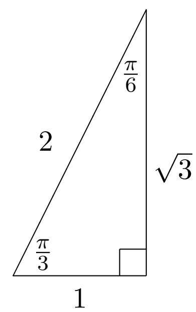
# Worksheet 4.6 Properties of Trigonometric functions

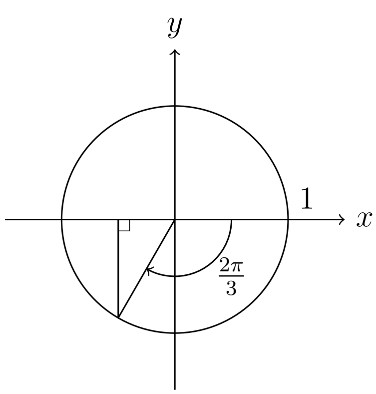
## Section 1 Review of Trigonometry

This section reviews some of the material covered in Worksheets 2.2, 3.3 and 3.4. The reader should be familiar with the trig ratios, using radians and working with exact values which arise from the following standard triangles.



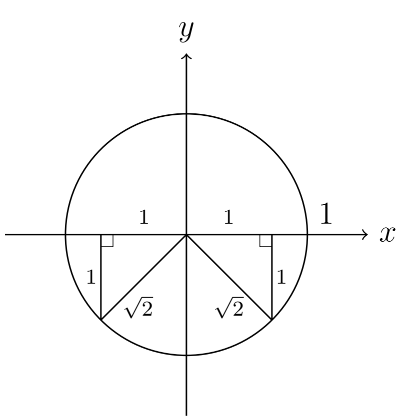




Example 1 : Find the exact value of $\tan \frac{-2\pi}{3}$.




- $-\frac{2\pi}{3}$ lies in the third quadrant and the angle made with the horizontal axis is $\frac{\pi}{3}$.
- tan is positive in the 3rd quadrant
- looking at the corresponding standard triangle in the third quadrant we see that $\tan(\frac{-2\pi}{3}) = +\sqrt{3}$

Example 2 : Find $\theta$ if $\sin\theta = -\frac{1}{\sqrt{2}}$ and $0 \leq \theta \leq 2\pi$.



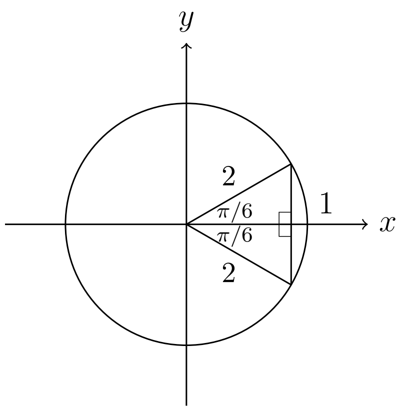

- since $\sin\theta$ is negative it must lie in the third quadrant o fourth quadrant
- looking at the standard triangle where $\sin\theta = \frac{1}{\sqrt{2}}$ in the third and fourth quadrant we see that

$$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

or

$$\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

Example 3 : Find $\theta$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $-\pi \le \theta \le \pi$.




- since $\cos\theta$ is positive it must lie in the first or fourth quadrant
- look at the standard triangles where $\cos\theta = \frac{\sqrt{3}}{2}$ in the first and fourth quadrants
- note that $-\pi \le \theta \le \pi$
- so $\theta = \frac{\pi}{6}$ or $\theta = -\frac{\pi}{6}$

Exercises:

1. Find the exact values of the following trig ratios.

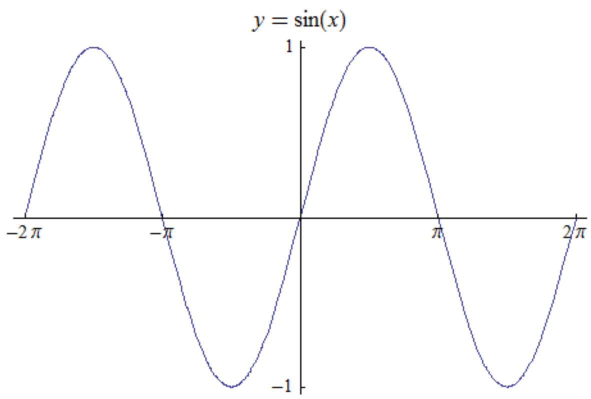
   (a) $\tan(\frac{5\pi}{3})$
   (b) $\sin(-\frac{10\pi}{3})$
   (c) $\cos(\frac{9\pi}{4})$
   (d) $\sin(\frac{34\pi}{6})$
   (e) $\sec(\frac{5\pi}{6})$
   (f) $\cot(-\frac{11\pi}{4})$

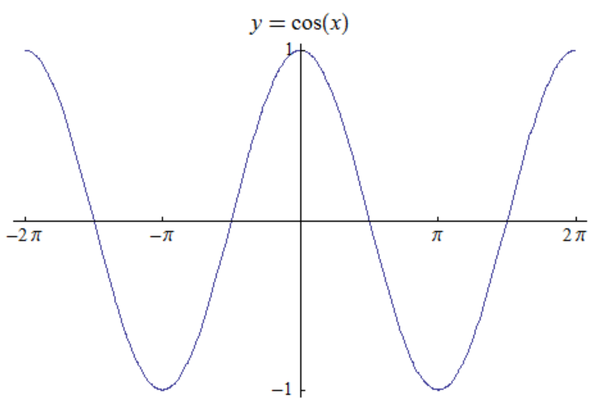
2. Find the value of $\theta$ in the following exercises.

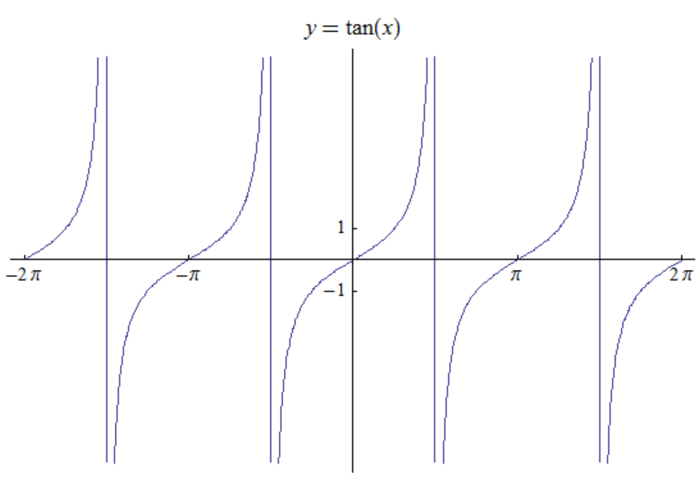
   (a) $\cos\theta = -\frac{\sqrt{3}}{2}$ where $0 \le \theta \le 2\pi$
   (b) $\tan\theta = \frac{1}{\sqrt{3}}$ where $0 \le \theta \le 2\pi$
   (c) $\sin\theta = -\frac{\sqrt{3}}{2}$ where $-\pi \le \theta \le \pi$
   (d) $\sec\theta = -\sqrt{2}$ where $0 \le \theta \le 4\pi$
   (e) $\csc\theta = 2$ where $\frac{\pi}{2} \le \theta \le 2\pi$
   (f) $\tan^2\theta = 1$ where $0 \le \theta \le 2\pi$

## Section 2 GRAPHS OF TRIGONOMETRIC FUNCTIONS

Recall the graphs of the trig functions described below for $-2\pi \leq x \leq 2\pi$.



We can see some properties of the trig functions from their graphs.

① These trig functions are periodic – they repeat themselves after a certain period.

- $\sin x$ and $\cos x$ both have period $2\pi$. i.e.

$$\sin x = \sin(x + 2\pi) \qquad \forall\, x \in \mathbb{R}$$
$$\cos x = \cos(x + 2\pi) \qquad \forall\, x \in \mathbb{R}$$

- $\tan x$ has period $\pi$. i.e.

$$\tan x = \tan(x + \pi) \qquad \forall\, x \in \mathbb{R}$$

② Note that $\sin x$ and $\cos x$ both lie between $-1$ and 1.

③ Note that $\tan x$ is undefined for $x = \frac{(2k-1)\pi}{2}$ when $k \in \mathbb{Z}$.

④ From the graphs we can see that

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

⑤ Since $\sin x$ and $\tan x$ are odd functions we have

$$\sin(-x) = -\sin x \qquad \forall\, x \in \mathbb{R}$$
$$\tan(-x) = -\tan x \qquad \forall\, x \in \mathbb{R}$$

Since $\cos x$ is an even function we have

$$\cos(-x) = \cos x \qquad \forall\, x \in \mathbb{R}$$

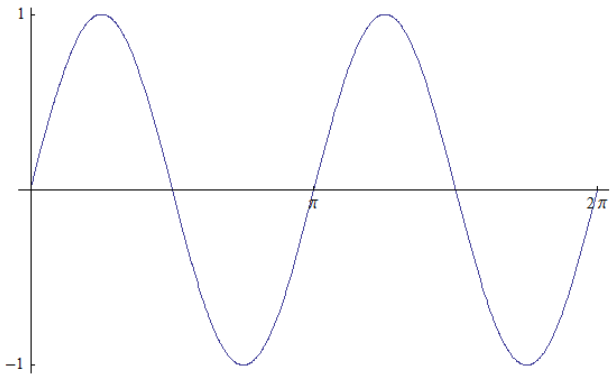
These graphs alter if we change either the period or amplitude, or if there is a phase shift. Consider the graph of $y = \sin x$. In general we can think of this as $y = A \sin n(x - a)$, where

- $A$ is the amplitude
- $n$ alters the period (period $= \frac{2\pi}{n}$)
- by subtracting $a$ from $x$, the graph shifts to the right by $a$.

So $y = \sin x$ has amplitude 1 and period $2\pi$.

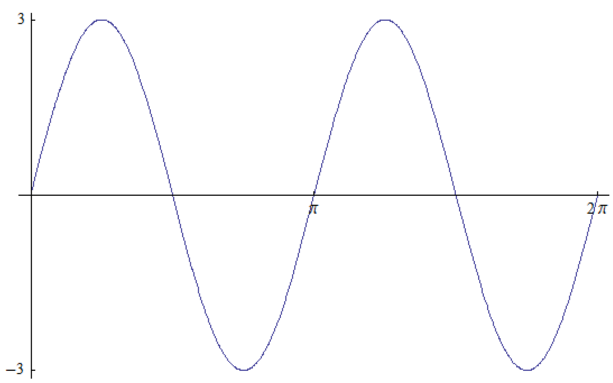
Example 1 : Sketch $y = \sin 2x$ for $0 \leq x \leq 2\pi$.

Here the period is now $\frac{2\pi}{2} = \pi$.



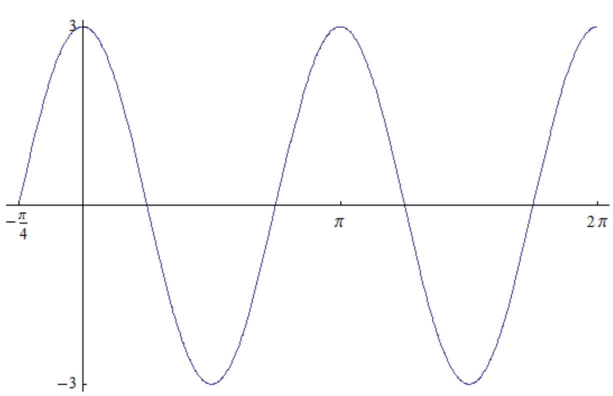

Example 2 : Sketch $y = 3\sin 2x$ for $0 \leq x \leq 2\pi$.

Here the period is still $\pi$ but the amplitude is now 3.




Example 3 : Sketch $y = 3\sin 2(x + \frac{\pi}{4})$ for $-\frac{\pi}{4} \leq x \leq 2\pi$.

The period is $\pi$, the amplitude is 3 and there is a phase shift. The graph shifts to the left by $\frac{\pi}{4}$.



Exercises:

1. Sketch the following graphs and state the period for each.
   (a) $y = \cos 4x$
   (b) $y = 2\cos 4(x - \frac{\pi}{3})$
   (c) $y = \tan 2(x + \frac{\pi}{2})$
   (d) $y = 1 + \sin(\frac{x-\pi}{3})$
   (e) $y = 2 - \cos(x + \frac{\pi}{6})$
   (f) $y = |\cos x|$
   (g) $y = \cos |x|$

2. Solve the following equations for $0 \leq x \leq 2\pi$.
   (a) $3\cos^2 x - \cos x = 0$
   (b) $2\sin^2 x + \sin x - 1 = 0$
   (c) $4\cos^3 x - 4\cos^2 x - 3\cos x + 3 = 0$
   (d) $\tan^2 x + 2\tan x + 1 = 0$

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## Section 3 Trigonometric identities

This section states and proves some common trig identities.

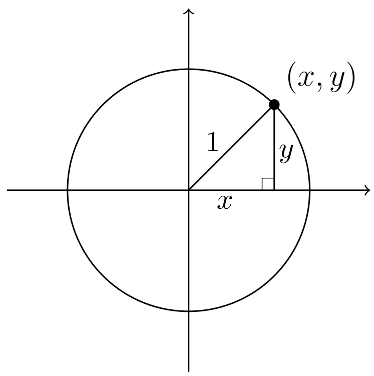
**Pythagorean Identities**

① $\cos^2\theta + \sin^2\theta = 1$

② $1 + \tan^2\theta = \sec^2\theta$

③ $1 + \cot^2\theta = \csc^2\theta$

Proof of ①: Consider a circle of radius 1 centred at the origin.




- Let $\theta$ be the angle measured anticlockwise for the positive $x$-axis.
- Using trig ratios we see that $x = \cos\theta$ and $y = \sin\theta$.
- By Pythagoras' Theorem $x^2 + y^2 = 1$. i.e. $\cos^2\theta + \sin^2\theta = 1$. □

Proof of ②: Divide both sides of identity ① by $\cos^2\theta$ and the result follows. □

Proof of ③: Divide both sides of identity ① by $\sin^2\theta$ and the result follows. □

**Sum and Difference Identities**

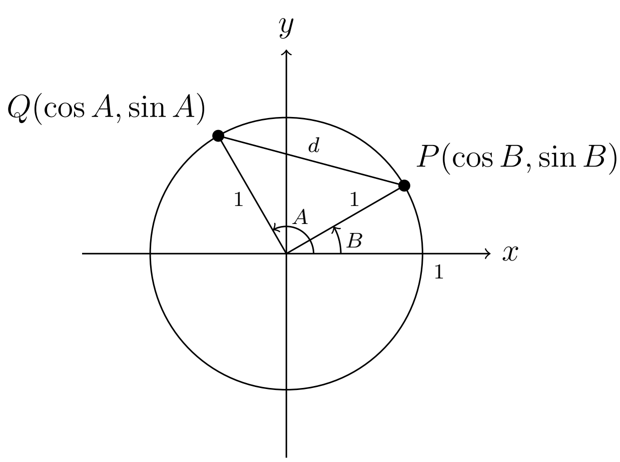
④ $\sin(A + B) = \sin A \cos B + \cos A \sin B$

⑤ $\sin(A - B) = \sin A \cos B - \cos A \sin B$

⑥ $\cos(A + B) = \cos A \cos B - \sin A \sin B$

⑦ $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Proof of ④ – ⑦: We first prove ⑦. Consider the following circle of radius 1 with angles $A$ and $B$ as shown.



Note that we can label point $P$ as $(\cos B, \sin B)$ and point $Q$ as $(\cos A, \sin A)$ by using trig ratios. We can calculate the distance $d$ using two methods.

Using the distance formula we see that

$$\begin{aligned} d^2 &= (\cos B - \cos A)^2 + (\sin B - \sin A)^2 \\ &= \cos^2 B - 2\cos A \cos B + \cos^2 A + \sin^2 B - 2\sin A \sin B + \sin^2 A \\ &= (\cos^2 B + \sin^2 B) + (\cos^2 A + \sin^2 A) - 2(\cos A \cos B + \sin A \sin B) \\ &= 2 - 2(\cos A \cos B + \sin A \sin B) \end{aligned}$$

Using the cosine rule we have

$$\begin{aligned} d^2 &= 1^2 + 1^2 - 2(1)(1)\cos(A - B) \\ &= 2 - 2\cos(A - B) \end{aligned}$$

Equating these we see that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

thus proving sum and difference identity ⑦.

We can use identity ⑦ to deduce the remaining identities. We have

$$\begin{aligned} \cos(A + B) &= \cos(A - (-B)) \\ &= \cos A \cos(-B) + \sin A \sin(-B) \\ &= \cos A \cos B - \sin A \sin B \end{aligned}$$

$$\begin{aligned} \sin(A + B) &= \cos\big(\tfrac{\pi}{2} - (A + B)\big) \\ &= \cos\big(\tfrac{\pi}{2} - A - B\big) \\ &= \cos\big(\tfrac{\pi}{2} - A\big)\cos B + \sin\big(\tfrac{\pi}{2} - A\big)\sin B \\ &= \sin A \cos B + \cos\big(\tfrac{\pi}{2} - \big(\tfrac{\pi}{2} - A\big)\big)\sin B \\ &= \sin A \cos B + \cos A \cos B \end{aligned}$$

$$\begin{aligned} \sin(A - B) &= \sin(A + (-B)) \\ &= \sin A \cos(-B) + \cos A \sin(-B) \\ &= \sin A \cos B - \cos A \sin B \end{aligned}$$

We have now established identities ④ – ⑥. □

**Double Angle Identities**

⑧ $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$= 2\cos^2\theta - 1$

⑨ $\sin 2\theta = 2\sin\theta\cos\theta$

Proof of ⑧ and ⑨: using the sum and difference identities we can prove the double angle identities. For instance,

$$\begin{aligned}\cos 2\theta &= \cos(\theta+\theta)\\ &\cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Replacing $\cos^2\theta$ by $1-\sin^2\theta$ (Pythagorean identity ①), we can see that $\cos 2\theta = 1 - 2\sin^2\theta$.

Replacing $\sin^2\theta$ by $1-\cos^2\theta$ (Pythagorean identity ①), we can see that $\cos 2\theta = 2\cos^2\theta - 1$.

We also have

$$\begin{aligned}\sin 2\theta &= \sin(\theta+\theta)\\ &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ &= 2\sin\theta\cos\theta.\end{aligned}$$

We have now established identities ⑧ and ⑨. □

**Half Angle Identities**

⑩ $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$

⑪ $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$

To prove the half angle identities we begin by rearranging the double angle identities.

Proof of ⑩: We take the double angle identity $\cos 2\theta = 2\cos^2\theta - 1$ to obtain

$$\begin{aligned}2\cos^2\theta &= \cos 2\theta + 1\\ \text{i.e.}\quad \cos^2\theta &= \frac{\cos 2\theta + 1}{2}\\ \text{i.e.}\quad \cos^2\left(\frac{\theta}{2}\right) &= \frac{\cos\theta + 1}{2}\end{aligned}$$

Proof of ⑪: We take the double angle identity $\cos 2\theta = 1 - \sin^2\theta$ to obtain

$$2\sin^2\theta = 1 - \cos 2\theta$$
$$\text{i.e.} \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\text{i.e.} \quad \sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

We have now established identities ⑩ and ⑪. □

Example 1 : Simplify $2\cos x\cos 2x\sin 3x - 2\sin x\sin 2x\sin 3x$.

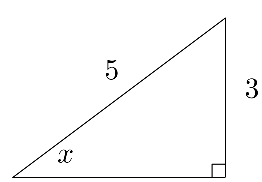
$$\begin{aligned} &2\cos x\cos 2x\sin 3x - 2\sin x\sin 2x\sin 3x \\ &\quad= 2\sin 3x(\cos x\cos 2x - \sin x\sin 2x) && \text{(factorise)} \\ &\quad= 2\sin 3x(\cos(x+2x)) && \text{(difference identity ⑥)} \\ &\quad= 2\sin 3x\cos 3x \\ &\quad= \sin 2(3x) && \text{(double angle identity ⑨)} \\ &\quad= \sin 6x \end{aligned}$$

Example 2 : Show that $2\cos^2 2x - \cos^2 x - \sin^2 x = \cos 4x$.

$$\begin{aligned} 2\cos^2 2x - \cos^2 x - \sin^2 x &= 2\cos^2 2x - (\cos^2 x + \sin^2 x) \\ &= 2\cos^2 2x - 1 && \text{(Pythagorean identity ①)} \\ &= \cos 2(2x) && \text{(double angle identity ⑧)} \\ &= \cos 4x \end{aligned}$$

Example 3 : Given $\sin x = \frac{3}{5}$ for $\frac{\pi}{2} \le x \le \pi$, find: (i) $\cos x$ and (ii) $\sin 2x$.

Since $\frac{\pi}{2} \le x \le \pi$, we know $x$ lies in the second quadrant. Also, since $\sin x = \frac{3}{5}$, we can form the following right angled triangle.




Using Pythagoras we see that the horizontal side is 4. We must have $\cos x = -\frac{4}{5}$, since cosine is negative in the second quadrant. So

$$\begin{aligned} \sin 2x &= 2\sin x\cos x \\ &= 2(\tfrac{3}{5})(-\tfrac{4}{5}) \\ &= -\tfrac{4}{25} \end{aligned}$$

Example 4 : Use the appropriate half angle identity to find the exact value of $\sin(\frac{\pi}{8})$.

The half angle identity for sine is ⑪ is

$$\begin{aligned} \sin^2\left(\frac{\theta}{2}\right) &= \frac{1-\cos\theta}{2} \\ \text{i.e.} \quad \sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{2}} \end{aligned}$$

Now,

$$\begin{aligned} \sin\left(\frac{\pi}{8}\right) &= \sin\left(\frac{\pi/4}{2}\right) \\ &= \pm\sqrt{\frac{1-\cos(\pi/4)}{2}} \\ &= \pm\sqrt{\frac{1-1/\sqrt{2}}{2}} \\ &= \pm\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \\ &= \pm\sqrt{\frac{2-\sqrt{2}}{4}} \\ &= \pm\frac{\sqrt{2-\sqrt{2}}}{2} \end{aligned}$$

Since $\frac{\pi}{8}$ lies in the first quadrant, where sine is positive, we must have

$$\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Exercises:

1. Simplify the following

   (a) $\cos 5x \sin x - \cos x \sin 5x$

   (b) $\dfrac{\sin^2 x + \cos^2 x + \tan^2 x}{\sec^2 x}$

   (c) $3 - 6\sin^2(x/8)$

   (d) $\dfrac{3\cot 2x \sin x \cos x}{\cos^2 x - \sin^2 x}$

   (e) $2\sin x \cos x - 4\sin^3 x \cos x$

   (f) $\frac{1}{2}\big(\cos(x/2) + 2\sin^2(x/4) - 1\big)$

2. Use the addition formulas to find the exact value of the following.

   (a) $\cos\left(\dfrac{7\pi}{12}\right)$

   (b) $\sin\left(\dfrac{14\pi}{12}\right)$

   (c) $\tan\left(\frac{7\pi}{6}\right)$

3. Use the half angle identities to calculate the exact value of the following.

   (a) $\cos\left(\frac{\pi}{8}\right)$

   (b) $\cos\left(\frac{\pi}{12}\right)$

   (c) $\sin\left(-\frac{\pi}{12}\right)$

4. Given $\tan x = \frac{5}{12}$ for $0 \leq x \leq \frac{\pi}{2}$, evaluate the following.

   (a) $\sin x$

   (b) $\cos x$

   (c) $\sin 2x$

   (d) $\cos 2x$

   (e) $\cos 3x$

# Exercises for Worksheet 4.6

1. Solve the following equations.

   (a) $\tan x = -\sqrt{3}$, $-\pi \le x \le \pi$

   (b) $\sin 2x = \frac{1}{2}$, $0 \le x \le 2\pi$

   (c) $4\sin^3 x - \sin x = 0$, $0 \le x \le 2\pi$

   (d) $\sec^2 x - 3\sec x + 2 = 0$, $0 \le x \le 2\pi$

2. Sketch the following curves.

   (a) $y = -2\cos\left(\frac{x}{3}\right)$

   (b) $y = 1 + \sin\left(x + \frac{\pi}{3}\right)$

   (c) $y = \tan 3\left(x - \frac{\pi}{6}\right)$

3. Prove the following.

   (a) $\cos 2x \sin x - \cos x \sin 2x = -\sin x$

   (b) $\dfrac{2\tan x - 2\sin^2 x \tan x}{\sin 2x} = 1$

   (c) $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \sin x}{\cos x}$

4. Suppose $\alpha$ lies in the third quadrant, $\beta$ lies in the fourth quadrant and $\sin\alpha = -\frac{4}{5}$ with $\cos\beta = \frac{12}{15}$. Find the following.

   (a) $\sin(\alpha + \beta)$

   (b) $\cos(\alpha + \beta)$

   (c) $\tan(\alpha + \beta)$

5. Suppose $\cos x = \frac{2}{3}$ where $\frac{3\pi}{2} \le x \le 2\pi$. Find the exact value of the following.

   (a) $\cos 3x$

   (b) $\sin 3x$

6. Use the half angle identity to find the exact value of $\sin(-\frac{3\pi}{8})$.

7. Use the appropriate identities to find the exact values of the following.

   (a) $\cos\left(\frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)$

   (b) $\sin\left(2\left(\frac{\pi}{6} + \frac{3\pi}{4}\right)\right)$

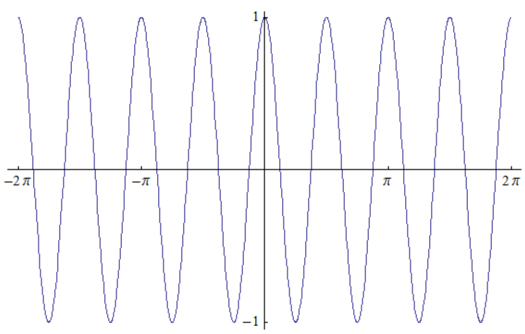
## Answers for Worksheet 4.6

Section 1

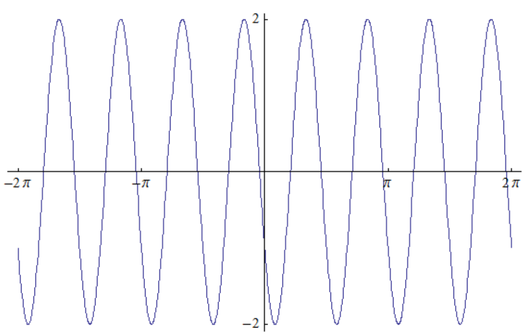
1. (a) $-\sqrt{3}$
   (b) $\dfrac{\sqrt{3}}{2}$
   (c) $\dfrac{1}{\sqrt{2}}$
   (d) $-\dfrac{\sqrt{3}}{2}$
   (e) $-\dfrac{2}{\sqrt{3}}$
   (f) 1
2. (a) $\dfrac{5\pi}{6}, \dfrac{7\pi}{6}$
   (b) $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$
   (c) $-\dfrac{\pi}{6}, -\dfrac{2\pi}{3}$
   (d) $\dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{11\pi}{4}, \dfrac{13\pi}{4}$
   (e) $\dfrac{5\pi}{6}$
   (f) $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

Section 2

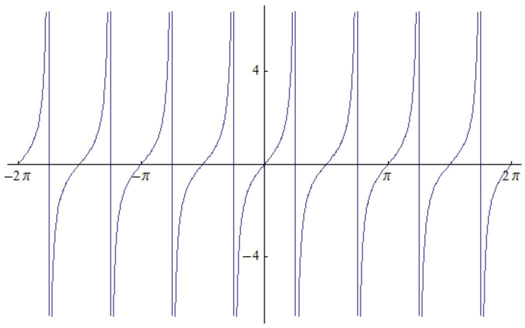
1. (a) period is $\dfrac{\pi}{2}$



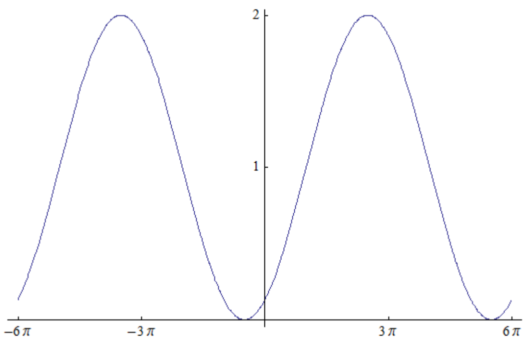

   (b) period is $\dfrac{\pi}{2}$



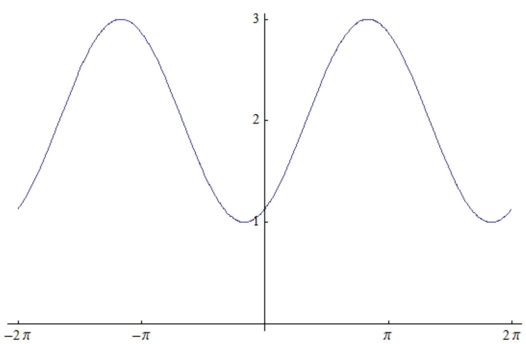

   (c) period is $\dfrac{\pi}{2}$



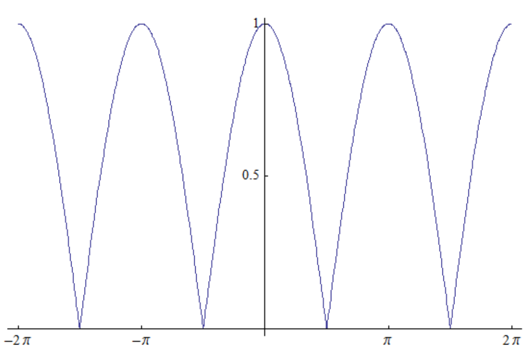

(d) period is $6\pi$



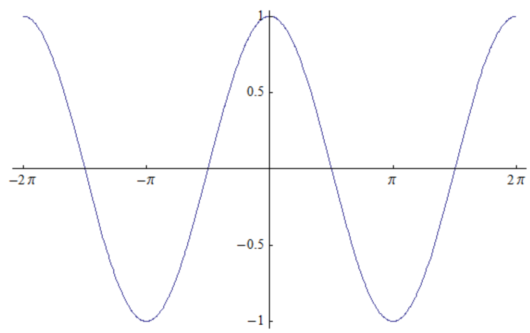

(e) period is $2\pi$




(f) period is $\pi$




(g) period is $2\pi$



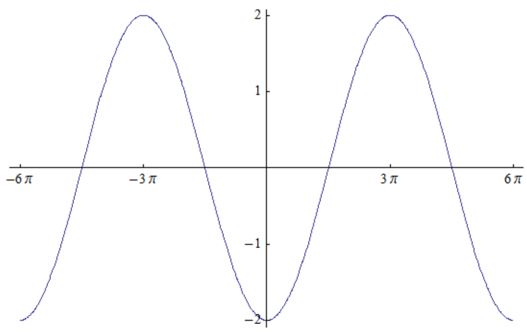

2. (a) $\frac{\pi}{2}, \frac{3\pi}{2}, 0.95, 5.34$
   (b) $\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$
   (c) $0, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
   (d) $\frac{3\pi}{4}, \frac{7\pi}{4}$

Section 3

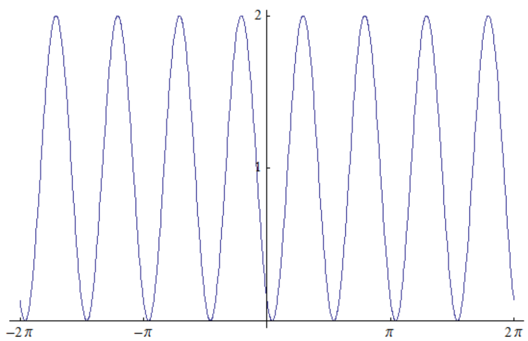
1. (a) $-\sin 4x$
   (b) $1$
   (c) $3\cos\left(\frac{x}{4}\right)$
   (d) $\frac{3}{2}$
   (e) $\frac{\sin 4x}{2}$
   (f) $0$
2. (a) $\frac{\sqrt{2}-\sqrt{6}}{4}$
   (b) $-\frac{1}{2}$
   (c) $\frac{1}{\sqrt{3}}$
3. (a) $\frac{\sqrt{2+\sqrt{2}}}{2}$
   (b) $\frac{\sqrt{2+\sqrt{3}}}{2}$
   (c) $-\frac{\sqrt{2-\sqrt{3}}}{2}$
4. (a) $\frac{5}{13}$
   (b) $\frac{12}{13}$
   (c) $\frac{120}{169}$
   (d) $\frac{828}{2197}$
   (e) $\frac{276}{715}$

**Exercises 4.6**

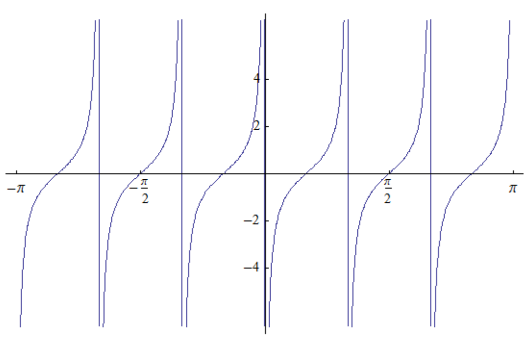
1. (a) $\frac{2\pi}{3}, -\frac{\pi}{3}$
   (b) $\frac{\pi}{12}, \frac{5\pi}{12}$
   (c) $0, \pi, 2\pi, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
   (d) $0, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$
2. (a) $y = -2\cos(\frac{x}{3})$




(b) $y = 1 + \sin(x + \frac{\pi}{3})$




(c) $y = \tan\big(2(x - \frac{\pi}{6})\big)$




3. Proofs only.

4. (a) $-\dfrac{7}{25}$ (b) $-\dfrac{24}{75}$ (c) $\dfrac{7}{24}$

5. (a) $-\dfrac{119}{54}$ (b) $-\dfrac{61\sqrt{5}}{72}$

6. $-\sqrt{2+\sqrt{2}}2$

7. (a) $\sqrt{\dfrac{4+\sqrt{2}+\sqrt{6}}{8}}$ (b) $-\dfrac{1}{2}$