Worksheet 4.4  Applications of Integration

Section 1  Movement

Recall that the derivative of a function tells us about its slope. What does the slope represent? It is the change in one variable with respect to the other variable. Say a line has a constant slope of 4; then for every 1 unit change in \( x \), there will be a 4 unit change in \( y \). Say we had a function that represented the movement of a car, so that the distance was plotted as a function of time. The change in distance over a small amount of time would represent the speed of the car. Thus in this case the slope of the function represents the speed of the car, and is given by \( \frac{dx}{dt} \). The rate of change in speed of the car is called acceleration and this is given by \( \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{d^2x}{dt^2} \).

So if we have a function \( x = f(t) \) that represents distance as a function of time, then \( \frac{dx}{dt} \) is the speed and \( \frac{d^2x}{dt^2} \) is the acceleration. Conversely, if we have a function that represents the velocity of a vehicle and we integrate it we get the distance travelled as a function of time.

The methods of integration and differentiation can be used to solve problems involving movement.

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Example 1: If the velocity of a particle is given by \( v = 3t^2 \), what is the distance travelled as a function of time? Since \( v = \frac{dx}{dt} \), where \( x \) is the distance, the anti derivative of \( v \) will tell us the distance.

\[
x = \frac{3t^3}{3} + c = t^3 + c
\]

Example 2: If the velocity of a particle is given as \( v = 3t^2 \) metres per second, what is the distance travelled between \( t = 0 \) and \( t = 2 \)? In example 1, we worked out that the distance travelled was \( x = t^3 + c \). Therefore

\[
\text{Distance} = (t^3 + c)_{0}^{2} = (2^3 + c) - (0^3 + c) = 8 \text{ metres}
\]

The particle covers a distance of 8 metres between \( t = 0 \) and \( t = 2 \).

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Section 2  Initial-Value Problems

Recall that, when working out anti derivative problems, there is a constant of integration that is undetermined (and which we have usually denoted by \( c \)). Initial-value problems ask us to
find anti derivatives which take on specific values at certain points so that we can determine
the value of the constant.

Example 1: If \( \frac{dx}{dt} = 5 \) and \( x = 9 \) when \( t = 0 \), what is \( x \) as a function of time? As \( \frac{dx}{dt} = 5 \), then \( x = 5t + c \). Using the information that when \( t = 0, x = 9 \) we can now write an equation to solve for \( c \):

\[
9 = 5 \times 0 + c
\]

which has the solution \( c = 9 \). The complete solution for \( x \) is \( x = 5t + 9 \).

Example 2: The acceleration of a car is given by \( a = 6t \) and the velocity is 2 when \( t = 0 \), and the distance from home is 1 when \( t = 0 \). What is the distance from home as a function of time? Acceleration is the derivative of velocity, so velocity is the anti derivative of acceleration. Then

\[
v = \int 6t \, dt = 3t^2 + c_1
\]

When \( t = 0, v = 2 \), so that we can write down an equation for \( c_1 \) and solve it: \( 2 = 0 + c_1 \). Therefore, \( c_1 = 2 \). The velocity is then \( v = 3t^2 + 2 \). Distance is the anti derivative of velocity, which gives

\[
x = \int (3t^2 + 2) \, dt = t^3 + 2t + c_2
\]

Using \( x = 1 \) when \( t = 0 \), we can write down an equation for \( c_2 \): \( 1 = 0 + 0 + c_2 \). Therefore \( c_2 = 1 \), and so the distance as a function of time is

\[
x = t^3 + 2t + 1
\]

Example 3: If \( \frac{d^2x}{dt^2} = 3 \), and when \( t = 0 \) we have \( \frac{dx}{dt} = 0 \) and \( x = 0 \), what is \( x \) as a function of \( t \)?

\[
\frac{d^2x}{dt^2} = 3 \\
\frac{dx}{dt} = 3t + c
\]

Using the information we are given for \( \frac{dx}{dt} \) at \( t = 0 \), we find \( 3 \times 0 + c = 0 \) so that \( c = 0 \). Then \( \frac{dx}{dt} = 3t \) for all \( t \). The anti derivative of this will give us \( x \):

\[
x = \frac{3t^2}{2} + c
\]

Using the information for \( x \) at \( t = 0 \), we find \( c = 0 \), so that \( x = \frac{3t^2}{2} \) for all \( t \).
Section 3  Application to Growth

Exponential functions are used to represent the growth and decay of populations and radioactive elements, among other things. We can use a general form of an equation for exponential growth or decay and we find a specific equation which uses initial values as in the application of integration to motion.

Exponential growth and decay is represented by the equation $P(t) = P(0)e^{kt}$ where $P(t)$ is the population at time $t$, $P(0)$ is the population at $t = 0$, and $k$ is some constant which depends on the population being looked at. A similar formula applies to the decay of radioactive material. $P(0)$ would then represent the amount of radioactive material at $t = 0$.

Example 1: What is $P(10)$ if $P(0) = 100$ and $k = 1$ given $P(t) = P(0)e^{kt}$.

\[ P(10) = P(0)e^{kt} = 100e^{10} \]

Example 2: If $P(10) = 1000$ and $P(0) = 100$, what is $k$ in the expression $P(t) = P(0)e^{kt}$? We put $t = 10$ into the equation for $P(t)$ and equate this to what we are given at $P(10)$.

\[ P(10) = 1000 = 100e^{10k} \]

This equation can be solved for $k$:

\[ \begin{align*}
1000 &= 100e^{10k} \\
10 &= e^{10k} \\
\log 10 &= 10k \\
k &= \frac{1}{10} \log 10
\end{align*} \]

Example 3: If the growth constant for a population of bees is $\frac{1}{10}$ and the initial population of a hive is 75, what is the population at time $t$?

\[ P(t) = P(0)e^{kt} = 75e^{\frac{1}{10}t} \]

Notice that if $k > 0$ the population is growing but if $k < 0$ the population is getting smaller.
Example 4: For what values of $k$ does the population $P(t) = P(0)e^{kt}$ remain constant? We need $P(t) = P(0)$ for all $t$. Then

$$\begin{align*}
P(t) &= P(0)e^{kt} \\
1 &= e^{kt}
\end{align*}$$

This is true when $k = 0$. 

Exercises for Worksheet 4.4

1. The derivatives of a function and one point on its graph are given. Find the function.

(a) \( \frac{dy}{dx} = x^3 + x^2 - 3; (1, 5) \)
(b) \( \frac{dy}{dx} = 2x(x + 1); (2, 0) \)
(c) \( y' = \cos x; (\frac{\pi}{6}, 4) \)
(d) \( y' = \frac{x}{\sqrt{10 - x^2}}; (1, 5) \)

2. (a) Find \( f(x) \) if its gradient function is \( 2x - 2 \) and \( f(1) = 4 \).

(b) The velocity \( v(t) \) of a particle moving in a straight line is given by \( v(t) = 12t^2 - 6t + 1 \), \( t \geq 0 \). Find its position coordinate \( s(t) = \int v(t) \, dt \) given that \( s(1) = 4 \).

(c) If \( \frac{dx}{dt} = kx \) and \( x = 10 \) when \( t = 0 \),
   i. Show that \( x = 10e^{kt} \).
   ii. Find \( k \) if \( x = 20 \) when \( t = 10 \).

(d) A radioactive substance decays according to the rule \( \frac{dM}{dt} = -0.2M \). If \( M = 5 \) when \( t = 0 \),
   i. Show that \( M = 5e^{-0.2t} \).
   ii. Find \( M \) when \( t = 5 \).

(e) A ship travelling at 10 metres per second is subjected to water resistance proportional to the speed. The engines are cut and the ship slows down according to the rule \( \frac{dv}{dt} = -kv \).
   i. Show that the velocity after \( t \) seconds is given by \( v = 10e^{-kt} \) metres per second.
   ii. If, after 20 seconds, \( v = 5 \text{m/s} \), find \( k \).

3. (a) A particle moves with constant acceleration of 5.8 metres/second squared. It starts with an initial velocity of 0.2m/s, and an initial position of 25m. Find the equation of motion of the particle given \( \int (\text{acceleration}) \, dt = \text{velocity} \), and \( \int (\text{velocity}) \, dt = \text{position} \).

(b) If the instantaneous rate of change of a population is \( 50t^2 - 100t^{\frac{3}{2}} \) (measured in individuals per year) and the initial population is 25000 then

   (a) What is the population after \( t \) years?
   (b) What is the population after 25 years?

(c) A particle moves along a straight line with an acceleration of \( a = 4 \sin \frac{\pi t}{2} \) m/s\(^2\). If the displacement at \( t = 0 \) is 0, and the initial velocity is \( -\frac{8}{5} \) m/s, find
   i. The acceleration after 2 seconds.
   ii. The velocity after 2 seconds.
   iii. The displacement after 2 seconds.
Answers for Worksheet 4.4

Exercises 4.4

1. (a) \( y = \frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x + \frac{89}{12} \)  
   (b) \( y = \frac{2}{3}x^3 + x^2 - \frac{28}{3} \)  
   (c) \( y = \sin x + \frac{7}{2} \)  
   (d) \( y = -(10 - x^2)^{\frac{1}{2}} + 8 \)

2. (a) 5  
   (b) 2  
   (c) \( k = \frac{\log 2}{10} \)  
   (d) \( \frac{5}{e} \)  
   (e) \( k = \frac{\log 2}{20} \)

3. (a) \( 2.9t^2 + 0.2t + 25 \)  
   (b) \( \frac{50t^3}{3} - 40t^{\frac{3}{2}} + 25000; \frac{841250}{3} \)  
   (c) i. 0  
   (ii. 0  
   iii. \( \frac{16}{\pi^2} - \frac{16}{\pi} \)