Worksheet 4.4 Applications of Integration

Section 1 Movement

Recall that the derivative of a function tells us about its slope. What does the slope represent? It is the change in one variable with respect to the other variable. Say a line has a constant slope of 4; then for every 1 unit change in x, there will be a 4 unit change in y. Say we had a function that represented the movement of a car, so that the distance was plotted as a function of time. The change in distance over a small amount of time would represent the speed of the car. Thus in this case the slope of the function represents the speed of the car, and is given by $\frac{dx}{dt}$. The rate of change in speed of the car is called acceleration and this is given by $\frac{d}{dt}(\frac{dx}{dt}) = \frac{d^2x}{dt^2}$.

So if we have a function $x = f(t)$ that represents distance as a function of time, then $\frac{dx}{dt}$ is the speed and $\frac{d^2x}{dt^2}$ is the acceleration. Conversely, if we have a function that represents the velocity of a vehicle and we integrate it we get the distance travelled as a function of time.

The methods of integration and differentiation can be used to solve problems involving movement.

Example 1 : If the velocity of a particle is given by $v = 3t^2$, what is the distance travelled as a function of time? Since $v = \frac{dx}{dt}$, where x is the distance, the anti derivative of v will tell us the distance.

$$x = \frac{3t^3}{3} + c = t^3 + c$$

Example 2 : If the velocity of a particle is given as $v = 3t^2$ metres per second, what is the distance travelled between $t = 0$ and $t = 2$? In example 1, we worked out that the distance travelled was $x = t^3 + c$. Therefore

$$\text{Distance } = (t^3 + c)\Big]_0^2 = (2^3 + c) - (0^3 + c) = 8 \text{ metres}$$

The particle covers a distance of 8 metres between $t = 0$ and $t = 2$.

Section 2 Initial-Value Problems

Recall that, when working out anti derivative problems, there is a constant of integration that is undetermined (and which we have usually denoted by c). Initial-value problems ask us to find anti derivatives which take on specific values at certain points so that we can determine the value of the constant.

Example 1 : If $\frac{dx}{dt} = 5$ and $x = 9$ when $t = 0$, what is x as a function of time? As $\frac{dx}{dt} = 5$, then $x = 5t + c$. Using the information that when $t = 0$, $x = 9$ we can now write an equation to solve for c:

$$9 = 5 \times 0 + c$$

which has the solution $c = 9$. The complete solution for x is $x = 5t + 9$.

Example 2 : The acceleration of a car is given by $a = 6t$ and the velocity is 2 when $t = 0$, and the distance from home is 1 when $t = 0$. What is the distance from home as a function of time? Acceleration is the derivative of velocity, so velocity is the anti derivative of acceleration. Then

$$v = \int 6t\, dt = 3t^2 + c_1$$

When $t = 0$, $v = 2$, so that we can write down an equation for c_1 and solve it: $2 = 0 + c_1$. Therefore, $c_1 = 2$. The velocity is then $v = 3t^2 + 2$. Distance is the anti derivative of velocity, which gives

$$x = \int (3t^2 + 2)\, dt = t^3 + 2t + c_2$$

Using $x = 1$ when $t = 0$, we can write down an equation for c_2: $1 = 0 + 0 + c_2$. Therefore $c_2 = 1$, and so the distance as a function of time is

$$x = t^3 + 2t + 1$$

Example 3 : If $\frac{d^2x}{dt^2} = 3$, and when $t = 0$ we have $\frac{dx}{dt} = 0$ and $x = 0$, what is x as a function of t?

$$\begin{aligned} \frac{d^2x}{dt^2} &= 3 \\ \frac{dx}{dt} &= 3t + c \end{aligned}$$

Using the information we are given for $\frac{dx}{dt}$ at $t = 0$, we find $3 \times 0 + c = 0$ so that $c = 0$. Then $\frac{dx}{dt} = 3t$ for all t. The anti derivative of this will give us x:

$$x = \frac{3t^2}{2} + c$$

Using the information for x at $t = 0$, we find $c = 0$, so that $x = \frac{3t^2}{2}$ for all t.

Section 3 APPLICATION TO GROWTH

Exponential functions are used to represent the growth and decay of populations and radioactive elements, among other things. We can use a general form of an equation for exponential growth or decay and we find a specific equation which uses initial values as in the application of integration to motion.

Exponential growth and decay is represented by the equation $P(t) = P(0)e^{kt}$ where $P(t)$ is the population at time t, $P(0)$ is the population at $t = 0$, and k is some constant which depends on the population being looked at. A similar formula applies to the decay of radioactive material. $P(0)$ would then represent the amount of radioactive material at $t = 0$.

Example 1 : What is $P(10)$ if $P(0) = 100$ and $k = 1$ given $P(t) = P(0)e^{kt}$.

$$\begin{aligned} P(10) &= P(0)e^{kt} \\ &= 100e^{10} \end{aligned}$$

Example 2 : If $P(10) = 1000$ and $P(0) = 100$, what is k in the expression $P(t) = P(0)e^{kt}$? We put $t = 10$ into the equation for $P(t)$ and equate this to what we are given at $P(10)$.

$$P(10) = 1000 = 100e^{k10}$$

This equation can be solved for k:

$$\begin{aligned} 1000 &= 100e^{10k} \\ 10 &= e^{10k} \\ \log 10 &= 10k \\ k &= \frac{1}{10}\log 10 \end{aligned}$$

Example 3 : If the growth constant for a population of bees is $\frac{1}{10}$ and the initial population of a hive is 75, what is the population at time t?

$$\begin{aligned} P(t) &= P(0)e^{kt} \\ &= 75e^{\frac{1}{10}t} \end{aligned}$$

Notice that if $k > 0$ the population is growing but if $k < 0$ the population is getting smaller.

Example 4 : For what values of k does the population $P(t) = P(0)e^{kt}$ remain constant? We need $P(t) = P(0)$ for all t. Then

$$\begin{aligned} P(t) &= P(0)e^{kt} \\ 1 &= e^{kt} \end{aligned}$$

This is true when $k = 0$.

Exercises for Worksheet 4.4

1. The derivatives of a function and one point on its graph are given. Find the function.

   (a) $\dfrac{dy}{dx} = x^3 + x^2 - 3$; $(1, 5)$

   (b) $\dfrac{dy}{dx} = 2x(x + 1)$; $(2, 0)$

   (c) $y' = \cos x$; $(\frac{\pi}{6}, 4)$

   (d) $y' = \dfrac{x}{\sqrt{10 - x^2}}$; $(1, 5)$

2. (a) Find $f(x)$ if its gradient function is $2x - 2$ and $f(1) = 4$.

   (b) The velocity $v(t)$ of a particle moving in a straight line is given by $v(t) = 12t^2 - 6t + 1$, $t \geq 0$. Find its position coordinate $s(t) = \int v(t)\,dt$ given that $s(1) = 4$.

   (c) If $\frac{dx}{dt} = kx$ and $x = 10$ when $t = 0$,

      i. Show that $x = 10e^{kt}$.

      ii. Find k if $x = 20$ when $t = 10$.

   (d) A radioactive substance decays according to the rule $\frac{dM}{dt} = -0.2M$. If $M = 5$ when $t = 0$,

      i. Show that $M = 5e^{-0.2t}$.

      ii. Find M when $t = 5$.

   (e) A ship travelling at 10 metres per second is subjected to water resistance proportional to the speed. The engines are cut and the ship slows down according to the rule $\frac{dv}{dt} = -kv$.

      i. Show that the velocity after t seconds is given by $v = 10e^{-kt}$ metres per second.

      ii. If, after 20 seconds, $v = 5$m/s, find k.

3. (a) A particle moves with constant acceleration of 5.8 metres/second squared. It starts with an initial velocity of 0.2m/s, and an initial position of 25m. Find the equation of motion of the particle given $\int$ (acceleration) dt = velocity, and $\int$ (velocity) dt = position.

   (b) If the instantaneous rate of change of a population is $50t^2 - 100t^{\frac{3}{2}}$ (measured in individuals per year) and the initial population is 25000 then

      (a) What is the population after t years?

      (b) What is the population after 25 years?

   (c) A particle moves along a straight line with an acceleration of $a = 4\sin\frac{\pi t}{2}$ m/s^2. If the displacement at $t = 0$ is 0, and the initial velocity is $-\frac{8}{\pi}$m/s, find

      i. The acceleration after 2 seconds.

      ii. The velocity after 2 seconds.

      iii. The displacement after 2 seconds.

Answers for Worksheet 4.4

Exercises 4.4

1. (a) $y = \frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x + \frac{89}{12}$
   (b) $y = \frac{2}{3}x^3 + x^2 - \frac{28}{3}$
   (c) $y = \sin x + \frac{7}{2}$
   (d) $y = -(10 - x^2)^{\frac{1}{2}} + 8$

2. (a) 5
   (b) 2
   (c) $k = \dfrac{\log 2}{10}$
   (d) $\dfrac{5}{e}$
   (e) $k = \dfrac{\log 2}{20}$

3. (a) $2.9t^2 + 0.2t + 25$
   (b) $\dfrac{50t^3}{3} - 40t^{\frac{3}{2}} + 25000$; $\dfrac{841250}{3}$
   (c) i. 0
      ii. 0
      iii. $\dfrac{16}{\pi^2} - \dfrac{16}{\pi}$