Worksheet 4.3 Integrating Special Functions

Section 1 Exponential and Logarithmic Functions

Recall from worksheet 3.10 that the derivative of e^x is e^x. It then follows that the anti derivative of e^x is e^x:

$$\int e^x \, dx = e^x + c$$

In worksheet 3.10 we also discussed the derivative of $e^{f(x)}$ which is $f'(x)e^{f(x)}$. It then follows that

$$\int f'(x)e^{f(x)} \, dx = e^{f(x)} + c$$

where $f(x)$ can be any function. There are other ways of doing such integrations, one of which is by substitution.

Example 1 : Evaluate the indefinite integral $\int 3e^{3x+2} \, dx$.

We recognize that $3 = \frac{d(3x+2)}{dx}$ so that the expression we are integrating has the form $f'(x)e^{f(x)}$. Then

$$\int 3e^{3x+2} \, dx = e^{3x+2} + c$$

Alternatively, we could do it by substitution: let $u = 3x + 2$. Then $du = 3dx$, and

$$\int 3e^{3x+2} \, dx = \int e^u \, du = e^u = e^{3x+2}$$

Note that the integral of the function e^{ax+b} (where a and b are constants) is given by

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + c$$

Example 2 : Find the area under the curve $y = e^{5x}$ between 0 and 2.

$$\begin{aligned} A &= \int_0^2 e^{5x} \, dx \\ &= \left. \frac{1}{5}e^{5x} \right]_0^2 \\ &= \frac{1}{5}e^{10} - \frac{1}{5}e^0 \\ &= \frac{1}{5}(e^{10} - 1) \end{aligned}$$

We used the property that for any real number x, $x^0 = 1$.

Recall that the derivative of $\log_e x$ is $\frac{1}{x}$. Then the anti derivative of $\frac{1}{x}$ is $\log_e x$. Notice that $\frac{1}{x} = x^{-1}$, and that if we had used the rules we have developed to find the anti derivatives of things like x^m, we would have the anti derivative of x^{-1} being $\frac{x^{-1+1}}{-1+1} = \frac{x^0}{0}$ which is not defined as we can not divide by zero. So we have the special rule for the anti derivative of $1/x$:

$$\int \frac{1}{x}\, dx = \log_e x + c$$

Recall that the derivative of $\log_e f(x)$ is $\frac{f'(x)}{f(x)}$. Then we have

$$\int \frac{f'(x)}{f(x)}\, dx = \log_e f(x) + c$$

<u>Example 3</u> : Evaluate the indefinite integral $\int \frac{5}{5x+2}\, dx$. This has the form $\int \frac{f'(x)}{f(x)}\, dx$ so we get

$$\int \frac{5}{5x+2}\, dx = \log_e(5x+2) + c$$

Note that when you need to integrate a function like $1/(ax+b)$ (where a and b are constants), then

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a}\int \frac{a}{ax+b}\, dx = \frac{1}{a}\log_e(ax+b) + c$$

<u>Example 4</u> : Find the area under the curve $f(x) = 1/(2x+3)$ between 3 and 11.

$$\begin{aligned}
A &= \int_3^{11} \frac{1}{2x+3}\, dx \\
&= \frac{1}{2}\log_e(2x+3)\Big]_3^{11} \\
&= \frac{1}{2}\log_e(2\times 11+3) - \frac{1}{2}\log_e(2\times 3+3) \\
&= \frac{1}{2}\log_e 25 - \frac{1}{2}\log_e 9 \\
&= \log_e(25)^{\frac{1}{2}} - \log_e(9)^{\frac{1}{2}} \\
&= \log_e \frac{5}{3}
\end{aligned}$$

Section 2 Integrating Trig Functions

To integrate trig functions we need to recall the derivatives of trig functions. We can then work out the anti derivatives of $\cos x$, $\sin x$, and $\sec^2 x$. For more complicated integrals we need special techniques that you will learn in first-year maths. The derivatives of the trig functions are:

$$\begin{aligned} g(x) &= \sin(ax+b) & g'(x) &= a\cos(ax+b) \\ f(x) &= \cos(ax+b) & f'(x) &= -a\sin(ax+b) \\ h(x) &= \tan(ax+b) & h'(x) &= a\sec^2(ax+b) \end{aligned}$$

Example 1 : Evaluate the indefinite integral $\int \sin 3x \, dx$.

$$\int \sin 3x \, dx = \frac{-1}{3} \cos 3x + c$$

Note : A good way of checking your answers to indefinite integrals is to differentiate them. You should recover the function that you started with.

Example 2 : Find the area under the curve $y = \cos x$ between 0 and $\frac{\pi}{2}$.

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \cos x \, dx \\ &= \sin x\big]_0^{\frac{\pi}{2}} \\ &= \sin \frac{\pi}{2} - \sin 0 \\ &= 1 \text{ square units} \end{aligned}$$

Example 3 : Find $\int f(x) \, dx$ if $f(x) = -3\sin(3x+2)$.

$$\int -3\sin(3x+2) \, dx = \cos(3x+2) + c$$

Example 4 : What is the area under the curve $y = \sec^2 \frac{x}{2}$ between $\frac{\pi}{2}$ and 0?

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2}\, dx \\ &= \frac{1}{1/2} \tan \frac{x}{2} \Big]_0^{\frac{\pi}{2}} \\ &= 2 \tan \frac{x}{2} \Big]_0^{\frac{\pi}{2}} \\ &= 2 \tan \frac{\pi}{4} - 2 \tan 0 \\ &= 2 - 0 \\ &= 2 \quad \text{square units} \end{aligned}$$

Example 5 : Evaluate the indefinite integral $\int 5 \sec^2 5x\, dx$.

$$\int 5 \sec^2 5x\, dx = \tan 5x + c$$

Exercises for Worksheet 4.3

1. (a) Find the anti derivative of

   i. e^{-4x}
   ii. $\sqrt{e^x}$
   iii. $\dfrac{7-6x}{8+7x-3x^2}$
   iv. $\cos 2x$
   v. $\sec^2(5x-2)$
   vi. $\dfrac{1-x}{x^2}$

   (b) Evaluate

   i. $\displaystyle\int_0^{\frac{1}{2}} e^{2x}\,dx$
   ii. $\displaystyle\int_{-1}^{1} \frac{2x+1}{x^2+x+1}\,dx$
   iii. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$
   iv. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\cos x\,dx$

2. (a) Calculate the area under the curve $y = \frac{2}{x+3}$ from $x = 2$ to $x = 3$.

   (b) Calculate the area under the curve $y = e^{3x}$ from $x = 0$ to $x = 3$.

   (c) The area under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = b$ is 1 unit. What is b?

   (d) Find the points of intersection of the curve $y = \sin x$ with the line $y = \frac{1}{2}$ and hence find the area between the two curves (from one intersection to the next). There are two possible areas you can end up with; choose the one above $y = \frac{1}{2}$.

   (e) Show, by simple division, that $\dfrac{x+6}{x+2} = 1 + \dfrac{4}{x+2}$. Hence evaluate $\displaystyle\int \frac{x+6}{x+2}\,dx$.

Answers for Worksheet 4.3

Exercises 4.3

1. (a) i. $-\dfrac{e^{-4x}}{4} + C$
   ii. $2e^{x/2} + C$
   iii. $\log(8 + 7x - 3x^2) + C$
   iv. $\dfrac{\sin 2x}{2}$
   v. $\frac{1}{5}\tan(5x - 2)$
   vi. $-\dfrac{1}{x} - \log x$

   (b) i. $\dfrac{1}{2}(e - 1)$
   ii. $\log 3$
   iii. 1
   iv. $\dfrac{1}{3}$

2. (a) $2\log\left(\frac{6}{5}\right)$
   (b) $\dfrac{1}{3}(e^9 - 1)$
   (c) e
   (d) $\sqrt{3} - \dfrac{\pi}{3}$
   (e) $x + 4\log(x + 2) + C$