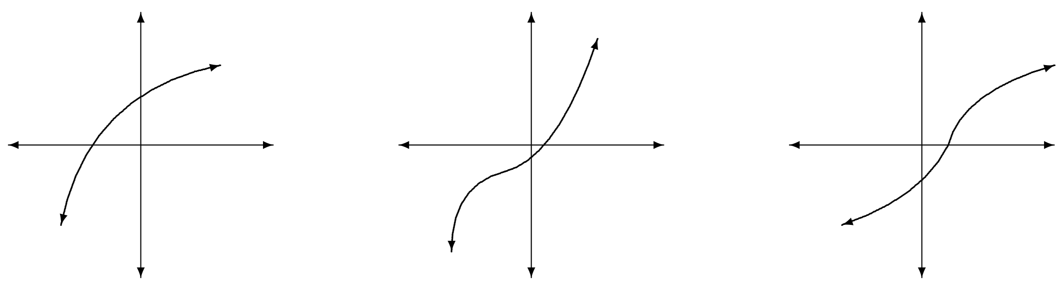
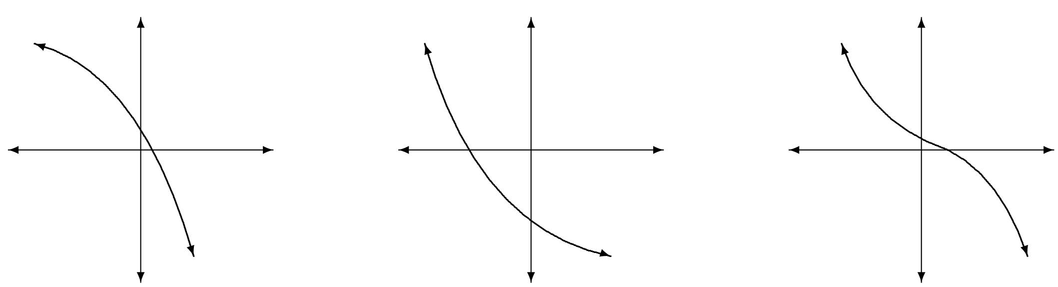
Worksheet 4.2 Introduction to Integration

Section 1 Left and Right Rectangles

A function f which has the property that if $b > a$ then $f(b) > f(a)$ is called monotonically increasing - as the input increases, then the output increases. The 'monotonically' part comes from the property that there are no maxima or minima. The slope of a monotonically increasing function will always be greater than or equal to zero, and it will only equal zero at a point of inflection. Here are some examples of the graphs of monotonically increasing functions:

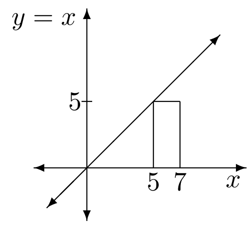


A monotonically decreasing function f is one which has the property that if $b > a$, then $f(b) < f(a)$. In other words, as the input gets larger, the output gets smaller. The slope of a monotonically decreasing function is always less than or equal to zero, and is only zero at a point of inflection. Here are some examples of the graphs of monotonically decreasing functions:



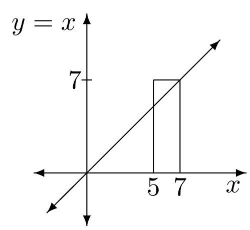
Sometimes it is important for us to be able to estimate the area under a curve, which might represent a quantity in which we are interested. For example, if we had a graph of a motorist's velocity as a function of time for a journey that lasted an hour, the area under the curve would represent the distance travelled over the journey. We now give a method that can be used to estimate the area under certain types of curves, namely those that are either monotonically increasing or decreasing. Note that all functions can be broken up into a sequence of parts, each of which is either (a) monotonically increasing or (b) monotonically decreasing or (c) horizontal. A vertical line is not a function as it does not have the property that each input value has one and only one output value. The estimation method involves splitting the area up into rectangles to give a lower and upper bound to the area under the curve.

Take the function $y = x$ as an example, and say we wish to know the area under the curve $y = x$ between $x = 5$ and $x = 7$. First we draw the graph:



If we draw a rectangle the height of the value of the function at $x = 5$ which stretches across to the same height above $x = 7$ we get the shaded region. This is called a left rectangle, as its height is given by the function value on the left hand side of the interval. Since $f(5) = 5$, the height of the rectangle is 5, and the width is $7 - 5 = 2$, so the area of the rectangle is $5 \times 2 = 10$.

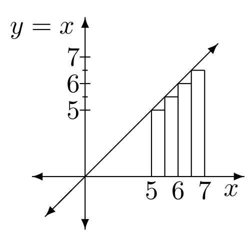
Now we draw the following diagram:

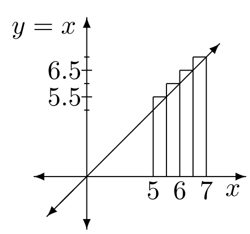


For the height of the rectangle, we use the value of the function at $x = 7$, which is $f(7) = 7$. This is called a right rectangle, and it has area $7 \times 2 = 14$. The area under the line $y = x$ between $x = 5$ and $x = 7$ must lie somewhere between 10 and 14 since the area of the right rectangle is bigger than the area under the line, and the area of the left rectangle is smaller than than the area under the line. Thus, if A is the area then,

$$10 < A < 14$$

We can get a closer approximation to the area under this line by breaking the interval into smaller pieces. Say we look at the function $y = x$ at every 1/2 unit, and add up the area of the rectangles formed by using as our intervals: [5,5.5],[5.5,6],[6,6.5], and [6.5,7]. The left rectangles are shown in figure 3, and the right rectangles in figure 4.





Note : The area of each rectangle is found by multiplying the base by the height. The area given by the left rectangles is the sum:

$$f(5)(5.5-5)+f(5.5)(6-5.5)+f(6)(6.5-6)+f(6.5)(7-6.5)=\frac{5}{2}+\frac{11}{4}+\frac{6}{2}+\frac{13}{4}=11\frac{1}{2}$$

The area given by the right rectangles is

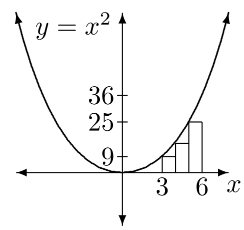
$$f(5.5)\frac{1}{2}+f(6)\frac{1}{2}+f(6.5)\frac{1}{2}+f(7)\frac{1}{2}=\frac{11}{4}+\frac{6}{2}+\frac{13}{4}+\frac{7}{2}=12\frac{1}{2}$$

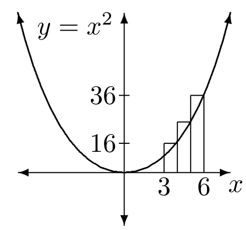
The lower and upper bounds on the area A are now given by:

$$11\frac{1}{2}<A<12\frac{1}{2}$$

By taking smaller and smaller intervals, we are going to bring the lower and upper bounds closer and closer together and so get a better approximation to the actual area.

Example 1 : Estimate the area under the curve $y=x^2$ from $x=3$ to $x=6$ by splitting the interval into 3 parts. Note that $y=x^2$ is monotonically increasing in the interval that we are interested in.





The left rectangles give a lower bound on the area:

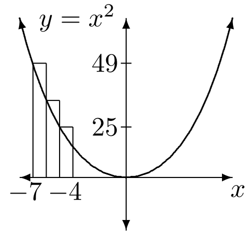
$$A_L=f(3)\times 1+f(4)\times 1+f(5)\times 1=9+16+25=50$$

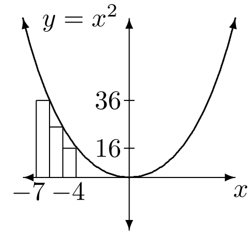
The right rectangles give an upper bound on the area:

$$A_R=f(4)\times 1+f(5)\times 1+f(6)\times 1=16+25+36=77$$

Therefore $50<A<77$. This is not a good approximation, but by taking smaller and smaller intervals, the error will be reduced.

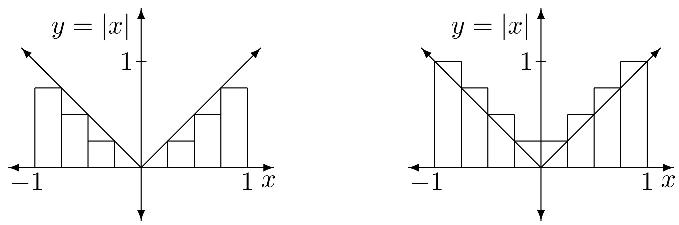
Example 2 : Estimate the area under the curve $y=x^2$ from $x=-7$ to $x=-4$ by splitting the interval into 3 parts. Note that $y=x^2$ is monotonically decreasing in the interval that we are interested in.





The left rectangles give $A_L = f(-7)\times 1 + f(-6)\times 1 + f(-5)\times 1 = 49+36+25 = 110$. The right rectangles give $A_R = f(-6)\times 1 + f(-5)\times 1 + f(-4)\times 1 = 36+25+16 = 77$. Notice that A_R is smaller than A_L. This is because now we are looking at a monotonically decreasing function, so A_R sets a lower bound and A_L an upper bound.

Example 3 : Find the area under the curve $y = |x|$ in the interval -1 to 1. Use 8 subintervals.



Since the interval is not monotonically increasing or decreasing, we need to split it up into pieces that are. So first we look at from -1 to 0, which yields the following:

$$\begin{aligned} A_L(-1,0) &= f(-1)\frac{1}{4} + f(-\frac{3}{4})\frac{1}{4} + f(-\frac{1}{2})\frac{1}{4} + f(-\frac{1}{4})\frac{1}{4} = \frac{4+3+2+1}{16} = \frac{5}{8} \\ A_R(-1,0) &= f(-\frac{3}{4})\frac{1}{4} + f(-\frac{1}{2})\frac{1}{4} + f(-\frac{1}{4})\frac{1}{4} + f(0)\frac{1}{4} = \frac{3+2+1}{16} = \frac{3}{8} \end{aligned}$$

The interval from 0 to 1 gives the estimates:

$$\begin{aligned} A_L(0,1) &= f(0)\frac{1}{4} + f(\frac{1}{4})\frac{1}{4} + f(\frac{1}{2})\frac{1}{4} + f(\frac{3}{4})\frac{1}{4} = \frac{3+2+1}{16} = \frac{3}{8} \\ A_R(0,1) &= f(\frac{1}{4})\frac{1}{4} + f(\frac{1}{2})\frac{1}{4} + f(\frac{3}{4})\frac{1}{4} + f(1)\frac{1}{4} = \frac{4+3+2+1}{16} = \frac{5}{8} \end{aligned}$$

To find a lower bound for the area in the given interval, we need to add the two lower bounds together, and similarly for the upper bound. Then the area we require is between $\frac{6}{8}$ and $\frac{10}{8}$. This problem could have been simplified by recognizing that $y = |x|$ is an even function. Then we would only have to double the lower and upper bounds for the area from $x = 0$ to $x = 1$.

Exercises:

1. Using the method described in this section estimate the area under the curve
   (a) $y = x^2$ between $x = 3$ and $x = 6$ using 3 rectangles and finding the upper and lower limits.
   (b) $y = 3x^2 + 1$ between $x = 0$ and $x = 4$ using 8 rectangles and finding the upper and lower limits.
   (c) $y = 4 - x^2$ between $x = -2$ and $x = 0$ using 4 rectangles and finding the upper and lower limits.

Section 2 Integrating Polynomials

Integration is a technique for finding, amoungst other things, the area under curves. Conceptually, it is like the method of left and right rectangles, but the number of subintervals that the interval of interest is broken up into is infinite, so we get an exact area where the lower and upper bounds are equal. We will not give the details of how one takes the limit of an infinite number of subintervals - we will just state some integration results.

Integration involves anti derivatives, so we will first look at these. The anti derivative of a function f is another function F such that

$$f(x) = F'(x)$$

Thus if $f(x)$ is the derivative of $F(x)$ then $F(x)$ is the anti derivative of $f(x)$. Worksheet 3.8 has an introduction to derivatives. Therefore we can reverse the rules that we had for polynomial differentiation to get anti derivative rules. Recall that if $f(x) = ax^n$, then $f'(x) = anx^{n-1}$. So if $g(x) = bx^m$ then an anti derivative $G(x)$ (such that $G'(x) = g(x)$) is given by

$$G(x) = \frac{b}{m+1}x^{m+1} \quad m \neq -1$$

We add one to the power of x then divide by the new power of x. Note that if

$$\begin{aligned} G(x) &= \frac{b}{m+1}x^{m+1} \\ \text{then} \quad G'(x) &= \frac{b}{m+1}(m+1)x^{m+1-1} \\ &= bx^m \\ &= g(x) \end{aligned}$$

which is what is required. Given $f'(x) = 2x$, then we could have $f(x) = x^2+1$ or $f(x) = x^2+3$ or $f(x) = x^2-4$; notice they differ by the constant term. To compensate for this - the property that the derivative of a constant is zero - we add a constant, usually denoted as c, to the anti derivative. We need more information to find distinct values of c.

Example 1 : Find the anti derivative $F(x)$ of the function $f(x) = 2x + 1$. Note $x^0 = 1$.

$$\begin{aligned} F(x) &= \frac{2x^{1+1}}{2} + \frac{1x^{0+1}}{1} + c \\ &= x^2 + x + c \end{aligned}$$

Example 2 : Find the anti derivative $G(x)$ of the function $g(x) = x^2 + 3x$.

$$\begin{aligned} G(x) &= \frac{x^{2+1}}{3} + \frac{3x^{1+1}}{2} + c \\ &= \frac{x^3}{3} + \frac{3x^2}{2} + c \end{aligned}$$

Example 3 : Find the anti derivative $H(x)$ of the function $h(x) = 5x^4 + 3x^2 + x + x^{-5} + 3$.

$$\begin{aligned} H(x) &= \frac{5x^{4+1}}{5} + \frac{3x^{2+1}}{3} + \frac{x^{1+1}}{2} + \frac{x^{-5+1}}{-4} + \frac{3x^{0+1}}{1} + c \\ &= x^5 + x^3 + \frac{x^2}{2} - \frac{x^{-4}}{4} + 3x + c \end{aligned}$$

Example 4 : Find the anti derivative $F(x)$ of $f(x) = x^{-2}$.

$$F(x) = \frac{x^{-2+1}}{-1} = \frac{x^{-1}}{-1} + c = \frac{-1}{x} + c$$

Example 5 : Find the anti derivative $F(x)$ of $f(x) = 1$.

$$F(x) = \frac{1x^{0+1}}{1} = x + c$$

Example 6 : Find the anti derivative of $f(x) = \frac{3}{x^2} + 4x + 5$. Call the anti derivative $F(x)$.

$$\begin{aligned} f(x) &= 3x^{-2} + 4x + 5 \\ F(x) &= \frac{3x^{-1}}{-1} + \frac{4x^2}{2} + 5x + C \\ &= -\frac{3}{x} + 2x^2 + 5x + C \end{aligned}$$

Exercises:

1. Find the anti derivative of each of the following functions

   (a) $6x^2 + 8x - 3$
   (b) $10x^4 - 3x^2 + 5$
   (c) $3x^4 - 6x^2 - 7$
   (d) $x + 3$
   (e) $x^3 - x^{-3} + 2x + 1$
   (f) $\frac{8}{x^3} - \frac{1}{x^2} + 3x + 4$
   (g) $4x^2 - \frac{7}{x^4} + 2$
   (h) $x^4 - 2x$
   (i) $63x^5 - 1$
   (j) $\frac{4}{x^3} - \frac{6}{x^2}$

Section 3 Integration

The area under the curve $y = f(x)$ between $x = a$ and $x = b$, where $f(x) \geq 0$ for $a \leq x \leq b$, is given by the formula

$$A = \int_a^b f(x)\, dx$$

This is read as the integral of the function $f(x)$ from a to b (where a is taken to be the smaller number). The integral can be evaluated using

$$\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a)$$

where $F(x)$ is an anti derivative of $f(x)$.

This is called a definite integral because we integrate between two given values $x = a$ and $x = b$ to obtain a single value. An indefinite integral is written as

$$\int f(x)\, dx = F(x)$$

where again $F(x)$ is an anti derivative of $f(x)$.

Example 1 : Calculate $\int 3x^2\, dx$.

$$\begin{aligned} \int 3x^2\, dx &= \frac{3x^{2+1}}{3} + c \\ &= \frac{3x^3}{3} + c \\ &= x^3 + c \end{aligned}$$

We have used the fact that $\int 3x^2\,dx = 3\int x^2\,dx$. In other words, we can 'pull' the 3 through the integral sign because the 3 is independent of the variable that we are integrating with respect to, which is x in this case. In general $\int af(x)\,dx = a\int f(x)\,dx$.

Note: An indefinite integral is the same as calculating the anti derivative.

Example 2 : Calculate $\int_0^1 (x+3)\,dx$.

$$\begin{aligned}
\int_0^1 (x+3)\,dx &= \left(\frac{x^{1+1}}{2} + \frac{3x^{0+1}}{1}\right)\Bigg]_0^1 \\
&= \left(\frac{x^2}{2} + 3x\right)\Bigg]_0^1 \\
&= \left(\frac{1^2}{2} + 3\times 1\right) - \left(\frac{0^2}{2} + 3\times 0\right) \\
&= 3\frac{1}{2}
\end{aligned}$$

Example 3 : Calculate the area under the curve $f(x) = x^2$ between $x = 3$ and $x = 6$. The area is given by

$$\begin{aligned}
A = \int_3^6 f(x)\,dx &= \int_3^6 x^2\,dx \\
&= \frac{x^3}{3}\Bigg]_3^6 \\
&= F(6) - F(3) \\
&= \frac{6^3}{3} - \frac{3^3}{3} \\
&= 63
\end{aligned}$$

Recall that, in example 1 in section 1, we found that the area was between 58 and 77.

Example 4 : Calculate the area under $f(x) = x$ between $x = 5$ and $x = 7$.

$$\begin{aligned}
A = \int_5^7 x\,dx &= \frac{x^2}{2}\Bigg]_5^7 \\
&= \frac{49}{2} - \frac{25}{2} \\
&= 12
\end{aligned}$$

See the example in section 1 for comparison.

Example 5 : Calculate the area under $f(x) = x^4 + x^2$ between $x = -1$ and $x = 0$.

$$\begin{aligned} A = \int_{-1}^{0} (x^4 + x^2)\, dx &= \left(\frac{x^5}{5} + \frac{x^3}{3}\right)\Bigg]_{-1}^{0} \\ &= \left(\frac{0^5}{5} + \frac{0^3}{3}\right) - \left(\frac{(-1)^5}{5} + \frac{(-1)^3}{3}\right) \\ &= 0 - \left(\frac{-1}{5} - \frac{1}{3}\right) \\ &= \frac{8}{15} \end{aligned}$$

Exercises:

1. Calculate the following integrals

(a) $\int_{-2}^{3} x + 7\, dx$

(b) $\int_{1}^{4} x^2 + 6\, dx$

(c) $\int_{3}^{5} x + 2\, dx$

(d) $\int_{0}^{4} x^2 + x - 1\, dx$

(e) $\int_{-1}^{2} 3x + 4\, dx$

(f) $\int_{0}^{2} 6 - 3x^2\, dx$

(g) $\int_{1}^{3} x^3 - 2x\, dx$

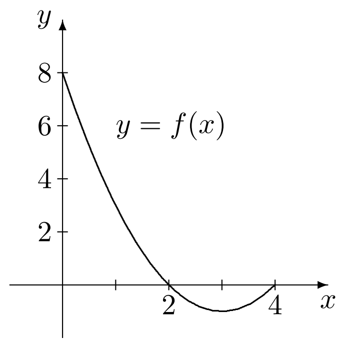
(h) $\int_{0}^{4} x + 2\, dx$

(i) $\int_{-3}^{-1} x^3 + x^2 - 6x\, dx$

(j) $\int_{4}^{6} x + 3\, dx$

Section 4 Integration Continued

As a further investigation of the area under a curve, we will look at the graph of the function $f(x) = x^2 - 6x + 8$.



We will find the area that is shaded. First find the shaded area between $x = 0$ and $x = 2$.

$$\begin{aligned}\int_0^2 x^2 - 6x + 8\,dx &= \left[\frac{x^3}{3} - 3x^2 + 8x\right]_0^2 \\ &= \left(\frac{8}{3} - 12 + 16\right) - (0 - 0 + 0) \\ &= 6\frac{2}{3}\end{aligned}$$

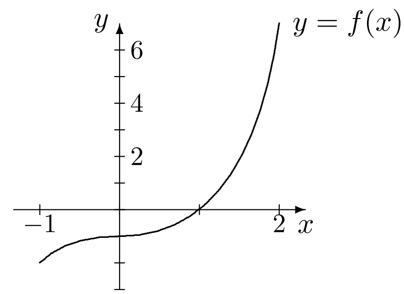
Now see what happens when we use the same method to find the shaded area between $x = 2$ and $x = 4$.

$$\begin{aligned}\int_2^4 x^2 - 6x + 8\,dx &= \left[\frac{x^3}{3} - 3x^2 + 8x\right]_2^4 \\ &= \left(\frac{64}{3} - 48 + 32\right) - \left(\frac{8}{3} - 12 + 16\right) \\ &= -1\frac{1}{3}\end{aligned}$$

An area cannot be negative. The negative sign indicates that the region is below the x axis – in this situation, the actual measure of the area is found by taking the absolute value of the integral. That is, the shaded area between $x = 2$ and $x = 4$ is

$$\left|\int_2^4 x^2 - 6x + 8\,dx\right| = \left|-1\frac{1}{3}\right| = 1\frac{1}{3}$$

Example 1 : Find the area bounded by the curve $y = x^3 - 1$, the x axis, and which lies between the lines $x = 0$ and $x = 1$. First draw the graph.



The required area is below the x axis, so

$$
\begin{aligned}
A &= \left|\int_0^1 x^3 - 1\,dx\right| \\
&= \left|\left[\frac{x^4}{4} - x\right]_0^1\right| \\
&= \left|\left(\frac{1}{4} - 1\right) - \left(\frac{0}{4} - 0\right)\right| \\
&= \left|-\frac{3}{4}\right| \\
&= \frac{3}{4}
\end{aligned}
$$

Example 2 : Find the area bound by the curve $y = x^3 - 1$, the x axis, and the lines $x = 0$ and $x = 3$.

Using the graph from the previous example as a guide, we see that the region from $x = 0$ to $x = 1$ is below the axis, and the region from $x = 1$ to $x = 3$ is above the x axis. So the area we want is

$$
\begin{aligned}
A &= \left|\int_0^1 x^3 - 1\,dx\right| + \int_1^3 x^3 - 1\,dx \\
&= \left|\left[\frac{x^4}{4} - x\right]_0^1\right| + \left[\frac{x^4}{4} - x\right]_1^3 \\
&= \left|\left(\frac{1}{4} - 1\right) - \left(\frac{0}{4} - 0\right)\right| + \left(\frac{81}{4} - 3\right) - \left(\frac{1}{4} - 1\right) \\
&= \left|-\frac{3}{4}\right| + 18 \\
&= 18\frac{3}{4}
\end{aligned}
$$

Exercises for Worksheet 4.2

1. (a) Use the method of left and right rectangles to find upper and lower bounds for the following functions and integration limits:

   i. $y = \sqrt{x}$ between $x = 0$ and $x = 1$ using 5 subdivisions.
   ii. $y = \frac{1}{x}$ between $x = 1$ and $x = 2$ using 10 subdivisions.

   (b) Find the anti derivative of the following functions:

   i. $f(x) = 1 + x + x^2$ ii. $g(x) = x^{\frac{1}{2}}$ iii. $h(x) = \frac{4}{x^3}$

   (c) Evaluate the following definite integrals:

   i. $\int_0^4 7x\,dx$ ii. $\int_0^1 (1 - y^2)\,dy$ iii. $\int_1^2 3t^2\,dt$

2. (a) By using rectangles of width 1, find the area under $y = [x]$ between $x = 0$ and $x = 5$ where $[x]$ is the 'greatest integer' function e.g. $[3.9] = 3$, $[4.1] = 4$.

   (b) Is the function in (i) monotonically increasing?

   (c) Which is greater, $\int_1^2 x\,dx$ or $\int_1^2 \sqrt{x}\,dx$?

   (d) Calculate the area of the region bounded by the graph of $f(x) = (x-2)^2$, the x-axis, and between $x = 2$ and $x = 3$.

   (e) Calculate the area bounded by the curve $y = x^2(3 - x)$ and the x-axis.

   (f) If $\displaystyle\int_{-1}^{a} x\,dx = 0$, evaluate a.

   (g) If $c\displaystyle\int_{-2}^{2} (x - 5)\,dx = 1$, evaluate c.

3. (a) Calculate the area bound by the curves $f(x) = \frac{x^2}{4} - 2$ and $g(x) = x + 1$. (Hint: Find the points of intersection of the two curves, and calculate both areas.)

   (b) Show, by integration, that the area of a unit square is:

   (a) Bisected by the line $y = x$.
   (b) Trisected by the curves $y = x^2$ and $y = \sqrt{x}$.

   (c) The marginal revenue, MR, that a manufacturer receives for his goods is given by $MR = \frac{dR}{dq} = 100 - 0.03q$. Find the total revenue function $R(q)$.

   (d) The density curve of a 10-metre beam is given by $\rho(x) = 3x + 2x^2 - x^{\frac{3}{2}}$ where x is the distance measured from one edge of the beam. The mass of the beam is calculated to be the area under the curve $\rho(x)$ between 0 and x. Find the mass of the beam.

Answers for Worksheet 4.2

Section 1

1. (a) $\frac{2480}{27}$; $\frac{1328}{27}$ (b) $\frac{161}{2}$; $\frac{113}{2}$ (c) $\frac{25}{4}$; $\frac{17}{4}$

Section 2

1. (a) $2x^3 + 4x^2 - 3x + C$
   (b) $2x^5 - x^3 + 5x + C$
   (c) $\frac{3x^5}{5} - 2x^3 - 7x + C$
   (d) $\frac{x^2}{2} + 3x + C$
   (e) $\frac{x^4}{4} + \frac{1}{2x^2} + x^2 + x + C$
   (f) $-\frac{4}{x^2} + \frac{1}{x} + \frac{3x^2}{2} + 4x + C$
   (g) $\frac{4x^3}{3} + \frac{7}{3x^3} + 2x + C$
   (h) $\frac{x^5}{5} - x^2 + C$
   (i) $\frac{63x^6}{6} - x + C$
   (j) $-\frac{2}{x^2} + \frac{6}{x} + C$

Section 3

1. (a) 75/2 (b) 39 (c) 12 (d) 76/3 (e) 33/2 (f) 4 (g) 12 (h) 16 (i) 38/3 (j) 16

Exercises 4.2

1. (a) i. The upper limit is $0.2 \times \left(\sqrt{0.2} + \sqrt{0.4} + \sqrt{0.6} + \sqrt{0.8} + 1\right)$.
   The lower limit is $0.2 \times \left(\sqrt{0.2} + \sqrt{0.4} + \sqrt{0.6} + \sqrt{0.8}\right)$.

   ii. The upper limit is $\frac{1}{10}\sum_{k=0}^{9}\frac{1}{1+\frac{k}{10}}$. The lower limit is $\frac{1}{10}\sum_{k=1}^{10}\frac{1}{1+\frac{k}{10}}$.

   (b) i. $x + \frac{x^2}{2} + \frac{x^3}{3} + C$ ii. $\frac{2x^{3/2}}{3} + C$ iii. $-\frac{2}{x^2} + C$

   (c)

i. 56
ii. 2/3
iii. 7

2. (a) 10
   (b) No
   (c) $\int_1^2 x\,dx$
   (d) 1/3
   (e) 27/4
   (f) 1
   (g) -1/20

3. (a) $-10/3$
   (c) $R = 100q - 0.015q^2 + C$
   (d) $\dfrac{3x^2}{2} + \dfrac{2x^3}{3} - \dfrac{2x^{5/2}}{5}$