Worksheet 4.1 More Differentiation

Section 1 The chain rule

In the last worksheet, you were shown how to find the derivative of functions like $e^{f(x)}$ and $\sin g(x)$. This section gives a method of differentiating those functions which are what we call composite functions. The method is called the chain rule. The chain rule allows us to differentiate composite functions. Composite functions are functions of functions, and can be written as

$$g(x) = f(u(x))$$

So if $u(x) = x^2$ and $f(u) = \cos u$, then

$$f(u(x)) = \cos x^2$$

The derivative of such functions is given by the following rule:

$$g'(x) = \frac{du(x)}{dx} \times \frac{df}{du}$$

So for our example of $g(x) = f(u(x)) = \cos x^2$ we have

$$\frac{df}{du} = -\sin u = -\sin x^2 \quad \text{and} \quad \frac{dg}{dx} = (2x) \times (-\sin x^2)$$

The trick is working out which function is the f and which is the u – it is what you do to the input first.

Example 1 : Differentiate e^{5x^2}. Let $u(x) = 5x^2$ and $f(u) = e^u$. If $g(x) = f(u(x))$ then

$$g'(x) = u'(x) \times \frac{df}{du}$$

We have $\frac{du}{dx} = 10x$ and $\frac{df}{du} = e^u = e^{5x^2}$ so that

$$g'(x) = 10xe^{5x^2}$$

Example 2 : Differentiate $g(x) = \sin(e^x)$. We let $u(x) = e^x$ and $f(u) = \sin u$. Then

$$\begin{aligned} u'(x) &= e^x \\ \frac{df}{du} &= \cos u = \cos e^x \\ \frac{dg}{dx} &= u'(x) \times \frac{df}{du} \\ &= e^x \cos(e^x) \end{aligned}$$

Example 3 : Differentiate $y = (6x^2+3)^4$. We let $u(x) = 6x^2+3$ and $f(u) = u^4$. Then

$$\begin{aligned} u'(x) &= 12x \\ \frac{df}{du} &= 4u^3 = 4(6x^2+3)^3 \\ \frac{dy}{dx} &= u'(x) \times f'(u) \\ &= 12x \times 4(6x^2+3)^3 \end{aligned}$$

Example 4 : Differentiate $y = (3x+2)^4$. Let $u(x) = 3x+2$. Then

$$\frac{dy}{dx} = 3 \times 4 \times (3x+2)^3 = 12(3x+2)^3.$$

Exercises:

1. Differentiate the following with respect to x.

   (a) $\sin 3x$
   (b) $\tan(-2x)$
   (c) $\cos 6x^2$
   (d) $(4x+5)^5$
   (e) $(6x-1)^3$
   (f) $(3x^2+1)^4$
   (g) e^{4x}
   (h) $7e^{2x}$
   (i) $e^{\sin x}$
   (j) $e^{\cos x}$
   (k) $(6-2x)^3$
   (l) $(7-x)^4$

Section 2 The product rule

The product rule gives us a method of working out the derivative of a function which can be written as the product of functions. Examples of such functions are $x^2 \sin x$, $5x \log x$, and $e^x \cos x$. These functions all have the general form

$$\begin{aligned} h(x) &= f(x)g(x) \quad \text{or in simpler terms} \\ h &= fg \end{aligned}$$

For functions that are written in this form, the product rule says:

$$\frac{dh}{dx} = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx} \quad \text{or} \quad h' = f'g + fg'$$

When first working with the product rule, it is wise to write down all the steps in the calculation to avoid any confusion.

Example 1 : Differentiate $h(x) = x^2 \sin x$.
Let $f(x) = x^2$ and $g(x) = \sin x$. Then $f'(x) = 2x$ and $g'(x) = \cos x$, which gives

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 2x \sin x + x^2 \cos x \end{aligned}$$

Note that in terms such as $\cos x \times x^2$, it is less ambiguous to write $x^2 \cos x$ to make it clear that we are not taking the cos of the x^2 term.

Example 2 : Differentiate $h(x) = 5x \log x$.
Let $f(x) = 5x$ and $g(x) = \log x$ so that $f'(x) = 5$ and $g'(x) = \frac{1}{x}$. Then

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 5 \log x + 5x \times \frac{1}{x} \\ &= 5 \log x + 5 \end{aligned}$$

Example 3 : Differentiate $p(x) = e^x \cos x$.
Let $a(x) = e^x$ and $b(x) = \cos x$. Then $a'(x) = e^x$ and $b'(x) = -\sin x$, so that

$$\begin{aligned} p'(x) &= a'(x)b(x) + a(x)b'(x) \\ &= e^x \cos x + e^x(-\sin x) \\ &= e^x(\cos x - \sin x) \end{aligned}$$

Example 4 : Differentiate $h(x) = 3x^2 e^x$.
Let $f(x) = 3x^2$ and $g(x) = e^x$ so that $f'(x) = 6x$ and $g'(x) = e^x$. Then

$$\begin{aligned} h'(x) &= 6xe^x + 3x^2 e^x \\ &= 3x(2e^x + xe^x) \\ &= 3x(x+2)e^x \end{aligned}$$

Note that, when using the product rule, it makes no difference which part of the whole function we call $f(x)$ or $g(x)$ (so long as we are able to differentiate the f or g that we choose). So in example 4, we could have let $f(x) = 6e^x$ and $g(x) = x$ and the final result for $h'(x)$ would have been the same.

Exercises:

1. Differentiate the following with respect to x.

   (a) $x^2 \sin x$

   (b) $4xe^{3x}$

   (c) $x^2 e^{3x}$

   (d) $x \cos x$

   (e) $4x \log(2x+1)$

   (f) $x^2 \log(x+2)$

   (g) $x^2 e^{x^3}$

   (h) $(3x+1)(x+1)^3$

   (i) $3x(x+2)^3$

   (j) $(4x-1)e^{2x}$

   (k) $\sin x e^{2x}$

   (l) $\cos(2x)e^{4x}$

Section 3 The quotient rule

The quotient rule is the last rule for differentiation that will be discussed in these worksheets. The quotient rule is derived from the product rule and the chain rule; the derivation is given at the end of the worksheet for those that are interested. The quotient rule helps to differentiate functions like $\frac{e^{2x}}{x^2}$, $\frac{x^2}{\cos x}$ and $\frac{x^2+1}{x^3+3}$. The general form of such expressions is given by $k(x) = \frac{u(x)}{v(x)}$, and the quotient rule says that

$$\begin{aligned} \frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) &= \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \quad \text{or} \\ k' &= \frac{u'v - uv'}{v^2} \end{aligned}$$

It is a good idea to do some bookkeeping when using the quotient rule.

Example 1 : Differentiate $k(x) = \frac{e^{2x}}{x^2}$.
Let $u(x) = e^{2x}$ and $v(x) = x^2$. Then $u'(x) = 2e^{2x}$ and $v'(x) = 2x$, which gives

$$\begin{aligned} k'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \\ &= \frac{2e^{2x}x^2 - e^{2x}2x}{(x^2)^2} \\ &= \frac{2xe^{2x}(x-1)}{x^4} \\ &= \frac{2e^{2x}(x-1)}{x^3} \end{aligned}$$

Note that the choice of $u(x)$ and $v(x)$ are not interchangeable as in the product rule. Given the complicated appearance of the quotient rule, it is wise to be consistent and always let $u(x)$ be the numerator and $v(x)$ the denominator.

Example 2 : Differentiate $p(x) = \frac{x^2}{\cos x}$. Let $u(x) = x^2$ and $v(x) = \cos x$. Then $u'(x) = 2x$ and $v'(x) = -\sin x$, so that

$$\begin{aligned} p'(x) &= \frac{2x\cos x - x^2(-\sin x)}{(\cos x)^2} \\ &= \frac{2x\cos x + x^2 \sin x}{\cos^2 x} \\ &= \frac{x(2\cos x + x\sin x)}{\cos^2 x} \end{aligned}$$

Example 3 : Differentiate $p(x) = \frac{x^2+1}{x^3+3}$.
Let $u(x) = x^2 + 1$ and $v(x) = x^3 + 3$. Then $u'(x) = 2x$ and $v'(x) = 3x^2$, so that

$$\begin{aligned} p'(x) &= \frac{2x(x^3+3) - (x^2+1)3x^2}{(x^3+3)^2} \\ &= \frac{6x - 3x^2 - x^4}{(x^3+3)^2} \end{aligned}$$

We now derive the quotient rule from the product and chain rule; skip the derivation if you don't feel the need to know. Let

$$k(x) = \frac{u(x)}{v(x)} = u(x)(v(x))^{-1}$$

We now use the product rule and let $f(x) = u(x)$ and $g(x) = (v(x))^{-1}$. Then $f'(x) = u'(x)$ and the derivative of $g(x)$ is given by the chain rule:

$$\begin{aligned} g'(x) &= -1(v'(x))(v(x))^{-2} \\ &= \frac{-v'(x)}{(v(x))^2} \end{aligned}$$

Using the product rule on $k(x)$ (the thing we are trying to differentiate), we get

$$\begin{aligned} k'(x) &= u'(x)(v(x))^{-1} + u(x) \times \frac{-v'(x)}{(v(x))^2} \\ &= \frac{u'(x)}{v(x)} - \frac{u(x)v'(x)}{(v(x))^2} \\ &= \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \end{aligned}$$

This is the quotient rule.

Exercises:

1. Differentiate the following with respect to x.

(a) $\dfrac{x^3}{x^2+1}$

(b) $\dfrac{x^2+3}{x+1}$

(c) $\dfrac{x-1}{2x+3}$

(d) $\dfrac{e^{2x}}{x-3}$

(e) $\dfrac{\sin x}{x^2}$

(f) $\dfrac{\sin x}{\cos x}$

(g) $\dfrac{3x}{x^2-2}$

(h) $\dfrac{x+6}{x-4}$

(i) $\dfrac{6e^x}{x+5}$

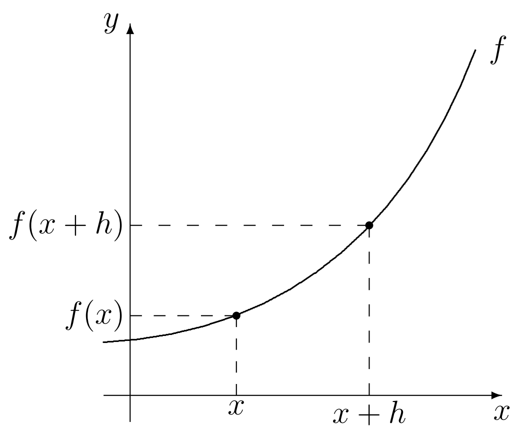
(j) $\dfrac{e^{2x}}{\sin x}$

Section 4 Equations of tangents and normals to curves

When the topic of differentiation was first introduced in section 1 of Worksheet 3.8, we said that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

This was motivated from this picture:

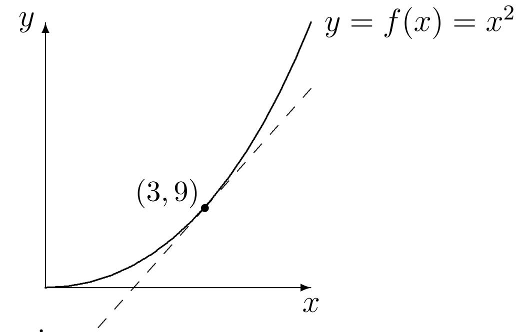


As $h \to 0$ the secant joining $(x, f(x))$ and $(x+h, f(x+h))$ becomes a better and better approximation to the tangent of f at the point $(x, f(x))$. We will now find the equation of this tangent. Recall that the tangent is just a straight line and it passes through the point $(x, f(x))$ on the curve. We have already found the equation of a straight line through a given point, say (x_1, y_1), with a given slope, say m – this was done in Worksheet 2.10. The equation of such a straight line is

$$y - y_1 = m(x - x_1).$$

<u>Example 1</u> : Find the equation of the tangent to the curve $y = x^2$ at the point $(3, 9)$.

A piece of the function is drawn as well as the tangent.



The derivative of the function is

$$\frac{dy}{dx} = 2x.$$

At the point $(3, 9)$, $\dfrac{dy}{dx} = 2 \times 3 = 6$ so that the slope of the tangent line is 6. Now, a point that lies on the tangent line is $(3, 9)$, so the equation of the tangent line is

$$\begin{aligned} y - 9 &= 6(x - 3) \\ y - 9 &= 6x - 18 \\ y &= 6x - 9 \end{aligned}$$

The equation of the tangent of $y = x^2$ at $(3, 9)$ is $y = 6x - 9$.

<u>Example 2</u> : Find the equation of the tangent to the curve $y = x^3 - x + 4$ at the point $(1, 4)$.

We will find the equation without drawing the graph. We have

$$\frac{dy}{dx} = 3x^2 - 1,$$

so the slope of the tangent at $x = 1$ is $3(1)^2 - 1 = 2$. A point that the tangent passes through is $(1, 4)$, so the equation must be given by

$$\begin{aligned} y - 4 &= 2(x - 1) \\ y - 4 &= 2x - 2 \\ y &= 2x + 2 \end{aligned}$$

The equation of the tangent of $y = x^3 - x + 4$ at $(1, 4)$ is $y = 2x + 2$.

Example 3 : Find the equation of the tangent to the curve $y = e^{2x}$ at the point $(3, e^6)$.

We have

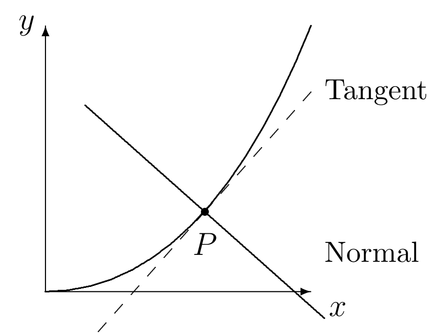
$$\frac{dy}{dx} = 2e^{2x},$$

so the slope of the tangent at $x = 3$ is $2e^6$. A point that the tangent passes through is $(3, e^6)$, so the equation must be given by

$$\begin{aligned} y - e^6 &= 2e^6(x-3) \\ y - e^6 &= 2e^6x - 6e^6 \\ y &= 2e^6x - 5e^6 \\ y &= e^6(2x-5) \end{aligned}$$

The equation of the tangent of $y = e^{2x}$ at $(3, e^6)$ is $y = e^6(2x-5)$.

The *normal* to a curve at a particular point is the straight line that passes through the point in question on the curve and is perpendicular to the tangent to the curve.



Example 4 : Find the equation of the normal to the curve $y = x^2$ at the point $(3, 9)$.

From example 1, the slope of the tangent is 6, so the gradient of the normal to the tangent is $-\frac{1}{6}$. (Recall that in section 3 of Worksheet 2.10 we said that if two lines are perpendicular, then the product of their slopes is -1.) So the equation of the normal at the point $(3, 9)$ is

$$\begin{aligned} y - 9 &= -\frac{1}{6}(x-3) \\ y - 9 &= -\frac{1}{6}x + \frac{1}{2} \\ y &= -\frac{1}{6}x + \frac{19}{2} \end{aligned}$$

Example 5 : Find the equations of the tangent and normal to the curve $y = x^3 - 5x + 6$ at $(-3, -6)$.

We have $\dfrac{dy}{dx} = 3x^2 - 5$, so the slope of the tangent when $x = -3$ is 22. The equation of the tangent is then given by

$$\begin{aligned} y - (-6) &= 22(x - (-3)) \\ y &= 22x + 60 \end{aligned}$$

The equation of the normal is given by

$$\begin{aligned} y - (-6) &= -\frac{1}{22}(x - (-3)) \\ y &= -\frac{1}{22}x - \frac{135}{22} \end{aligned}$$

Exercises:

1. Find the equation of the tangent to the curve
   (a) $y = x^2 - 4x + 6$ at the point $(-2, 18)$
   (b) $y = 6 - x^2$ when $x = 3$
   (c) $y = x^3 - 4x + 30$ when $x = -5$
2. Find the equation normal to the curve
   (a) $y = 8 - 3x^2$ at the point $(4, -40)$
   (b) $y = x^3 - 2x^2 + 6$ when $x = -1$
   (c) $y = 6/x$ at the point $(-2, -3)$.
3. Find the equation of the tangent to the curve $y = 3x^2 - 2x + 4$ at the point $(1, 5)$ and also find the point where the tangent cuts the x axis.

Exercises for Worksheet 4.1

1. Differentiate the following

   (a) $y = \frac{1}{x^2} - 6x + 4$

   (b) $y = xe^{2x}$

   (c) $y = \sin 2x - \cos 4x$

   (d) $y = (2x+1)^3(x+2)$

   (e) $y = 4x \sin x$

   (f) $y = \log(x^2 + 1)$

   (g) $y = x \log x$

   (h) $y = (\sin x)^2$

   (i) $y = e^x \sin x$

   (j) $y = \frac{6x+1}{x-4}$

   (k) $y = \frac{x^2}{x+3}$

   (l) $y = \frac{e^x}{x-2}$

2. (a) Find the equation of the tangent to the curve $y = x^2 \log x$ at the point (e, e^2).

   (b) If $f(x) = \sin 2x - \cos 4x$, find $f'(\frac{\pi}{4})$.

   (c) If $y = (x^2 - 1)(1 + x)$, show that $x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2x - 2 = 0$.

   (d) Find the turning point of the curve $y = x^2 + 3x - 4$ and state whether it is a maximum or minimum turning point.

Answers for Worksheet 4.1

Section 1

1. (a) $3\cos 3x$ (b) $-2\sec^2(-2x)$ (c) $-12x\sin 6x^2$ (d) $20(4x+5)^4$ (e) $18(6x-1)^2$ (f) $24x(3x^2+1)^3$ (g) $4e^{4x}$ (h) $14e^{2x}$ (i) $\cos x\, e^{\sin x}$ (j) $-\sin x\, e^{\cos x}$ (k) $-6(6-2x)^2$ (l) $-4(7-x)^3$

Section 2

1. (a) $2x\sin x + x^2\cos x$ (b) $4e^{3x}(1+3x)$ (c) $xe^{3x}(2+3x)$ (d) $\cos x - x\sin x$ (e) $4\log(2x+1)+\frac{8x}{2x+1}$ (f) $2x\log(x+2)+\frac{x^2}{x+2}$ (g) $xe^{x^3}(2+3x^3)$ (h) $(x+1)^2(12x+6)$ (i) $(x+2)^2(12x+6)$ (j) $2e^{2x}(4x+1)$ (k) $e^{2x}(\cos x+2\sin x)$ (l) $2e^{4x}(2\cos 2x-\sin 2x)$

Section 3

1. (a) $\dfrac{x^4+3x^2}{(x^2+1)^2}$ (b) $\dfrac{x^2+2x-3}{(x+1)^2}$ (c) $\dfrac{5}{(2x+3)^2}$ (d) $\dfrac{e^{2x}(2x-7)}{(x-3)^2}$ (e) $\dfrac{x\cos x-2\sin x}{x^3}$ (f) $\dfrac{1}{\cos^2 x}$ (g) $\dfrac{-3x^2-6}{(x^2-2)^2}$ (h) $\dfrac{-10}{(x-4)^2}$ (i) $\dfrac{e^x(6x+24)}{(x+5)^2}$ (j) $\dfrac{e^{2x}(2\sin x-\cos x)}{\sin^2 x}$

Section 4

1. (a) $y=-8x+2$ (b) $y=-6x+15$ (c) $y=71x+280$
2. (a) $y=\dfrac{x}{24}-\dfrac{241}{6}$ (b) $y=-\dfrac{x}{7}+\dfrac{20}{7}$ (c) $y=\dfrac{2x}{3}-\dfrac{5}{3}$
3. $y=4x+1;\qquad x=-1/4.$

Exercises 4.1

1. (a) $-\dfrac{2}{x^3} - 6$
   (b) $e^{2x}(1+2x)$
   (c) $2\cos 2x + 4\sin 4x$
   (d) $(2x+1)^2(5x+7)$
   (e) $4(\sin x + x\cos x)$
   (f) $\dfrac{2x}{x^2+1}$
   (g) $\log x + 1$
   (h) $2\sin x \cos x$
   (i) $e^x(\sin x + \cos x)$
   (j) $-\dfrac{25}{(x-4)^2}$
   (k) $\dfrac{x^2+6x}{(x+3)^2}$
   (l) $\dfrac{e^x(x-3)}{(x-2)^2}$

2. (a) $y = 3ex - 2e^2$
   (b) 0
   (d) Minimum at $(-3/2, -25/4)$.