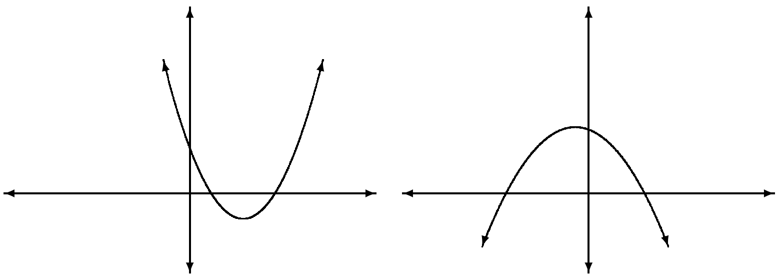
Worksheet 3.9 Further Differentiation

Section 1 Discriminant

Recall that the expression $ax^2 + bx + c$ is called a quadratic, or a polynomial of degree 2. The graph of a quadratic is called a parabola, and looks like one of the following:



They are symmetrical about a stationary point which is either a local minimum or maximum. Parabolas do not have points of inflection. In the quadratic $y = ax^2 + bx + c$, if the co-efficient of x^2 is greater than zero, the parabola is concave up; if a is negative, the parabola is concave down. The option $a = 0$ is precluded as this would result in a linear polynomial which is a straight line when graphed.

The quadratic formula found by solving $ax^2 + bx + c = 0$ is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The value under the square root sign, $b^2 - 4ac$, is called the discriminant, and we denote this by Δ. We can tell quite a lot about the curve $y = ax^2 + bx + c$ just by evaluating $\Delta = b^2 - 4ac$.

If $\Delta > 0$	then the graph will cut the x-axis in two places, i.e. there are two x-intercepts.
If $\Delta = 0$	the graph will touch the x-axis in one place only.
If $\Delta < 0$	the graph will sit wholly above or below the x-axis depending on the sign of a. There are no x-intercepts.

Using this information and the information gained from the derivative, we can sketch the graph of any quadratic.

Example 1 : Sketch the graph of $y = x^2 + 3x + 2$. We have $a = 1, b = 3$, and $c = 2$. Since $a > 0$, the stationary point is a minimum. Further, $b^2 - 4ac > 0$

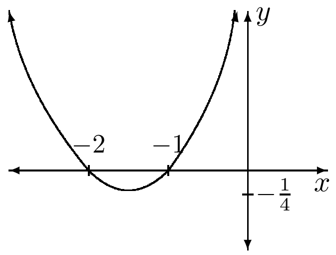
so there are two x-intercepts, which are found in this case through factorization: $x^2 + 3x + 2 = (x + 2)(x + 1)$. So $y = 0$ when $(x + 2)(x + 1) = 0$, or when $x = -1$ or -2. We find where the stationary point is by setting $\frac{dy}{dx} = 0$. This gives

$$\frac{dy}{dx} = 2x + 3 = 0$$

which has the solution $x = -\frac{3}{2}$. When $x = -\frac{3}{2}$,

$$y = (-\frac{3}{2})^2 + 3(-\frac{3}{2}) + 2 = -\frac{1}{4}$$

The stationary point is at $(-\frac{3}{2}, -\frac{1}{4})$. When $x = 0$, $y = 2$, so the y-intercept is 2. With all this information at our disposal, we can now draw the graph of $y = x^2 + 3x + 2$:



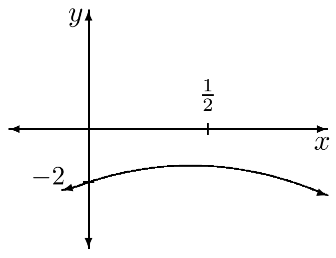
Example 2 : Sketch $y = -2 + x - x^2$. This can be rewritten as $y = -x^2 + x - 2$ so that $a = -1$, $b = 1$, and $c = -2$. We have $a < 0$ so the stationary point is a maximum. We find the x-coordinate of the stationary point by setting $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = -2x + 1 = 0$$

which has the solution $x = \frac{1}{2}$, and the corresponding y value is

$$y = -(\frac{1}{2})^2 + \frac{1}{2} - 2 = -\frac{7}{4}$$

The stationary point has the coordinates $(\frac{1}{2}, -\frac{7}{4})$. The discriminant is $b^2 - 4ac = -7 < 0$ so there are no x-intercepts. The y-intercept is given when $x = 0$, so $y = -2$. The graph of $y = -2 + x - x^2$ then looks like:



Exercises:

1. Sketch the graph of each of the following using the method studied in section 1.
   (a) $y = x^2 + 7x + 12$
   (b) $y = -x^2 + x + 6$
   (c) $y = x^2 + 4x + 5$
   (d) $y = x^2 - 16$
   (e) $y = 2x^2 - 5x - 3$

Section 2 Second Derivatives

Recall from the last worksheet the discussion on concavity. To determine whether a stationary point was a maximum, minimum, or a point of inflection, we looked at the changes in slope as we moved from one side of the stationary point to the other. There is an easier method of determining the concavity of a graph at any point (not just a critical point). The method is based upon the notion that concavity is a measure of the change of a slope.

As we viewed the derivative as a measure in the change of the height of a function we can view the derivative of the derivative as a change in the slope of the graph. The derivative of the derivative is called the second derivative, and it is denoted by one of the following:

$$f''(x),\ y'',\ \frac{d^2y}{dx^2},\ \text{or}\ \ D^2(f)$$

depending on how the function is defined.

Example 1 : Find the second derivative of $y = 5x^2 + 3$.

$$\begin{aligned} \frac{dy}{dx} &= 10x \\ \frac{d^2y}{dx^2} &= 10 \end{aligned}$$

So the second derivative of $y = 5x^2 + 3$ is 10.

Example 2 : Find the second derivative of $f(x) = x^3 + 3x^2 + 2x$.

$$\begin{aligned} f'(x) &= 3x^2 + 6x + 2 \\ f''(x) &= 6x + 6 \end{aligned}$$

The second derivative is a help with curve sketching as it tells about the concavity of the graph at any point - this is most useful at critical points.

If $f''(x) = 0$, the concavity is changing, so that the critical point is a point of inflection.

If $f''(x) > 0$, then the graph is concave up at x.

If $f''(x) < 0$, then the graph is concave down at x.

This means

(1) If $f'(x) = 0$ and $f''(x) > 0$ there is a minimum turning point at x.

(2) If $f'(x) = 0$ and $f''(x) < 0$ there is a maximum turning point at x.

Example 3 : For the function $f(x) = x^3 + 3x^2 + 3x$, find the stationary points and describe their important characteristics. We first find the solutions of $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 3x^2 + 6x + 3 \\ &= 3(x^2 + 2x + 1) \\ &= 3(x + 1)(x + 1) \\ &= 0 \end{aligned}$$

which has the solution $x = -1$. Also $f(-1) = (-1)^3 + 3(-1)^2 + 3(-1) = -1$. So the critical point is at $(-1, -1)$. Now,

$$f''(x) = 6x + 6$$

At $x = -1$, $f''(-1) = 6(-1) + 6 = 0$. Therefore the concavity is changing at $x = -1$, so the point $(-1, -1)$ is a point of inflection.

Example 4 : Find the stationary points of the function $f(x) = x^4 - x^2 + 1$, and describe their properties. We first find solutions of $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 4x^3 - 2x \\ &= 2x(2x^2 - 1) \end{aligned}$$

When $f'(x) = 0$, $2x(2x^2 - 1) = 0$ which has the solutions $x = 0$ and $x = \pm\frac{1}{\sqrt{2}}$. These are the x-coordinates of the critical points. The second derivative is $f''(x) = 12x^2 - 2$.

At $x = 0$, $f''(0) = -2 < 0$ so at $x = 0$ we have a local maximum.
At $x = \frac{1}{\sqrt{2}}$, $f''(\frac{1}{\sqrt{2}}) = 4 > 0$ so at $x = 0$ we have a local minimum.
At $x = -\frac{1}{\sqrt{2}}$, $f''(-\frac{1}{\sqrt{2}}) = 4 > 0$ so at $x = 0$ we have a local minimum.

Exercises:

1. Find the stationary points of each of the following and describe their important characteristics.

   (a) $f(x) = x^3 - 12x - 4$

   (b) $f(x) = 2x^4 - x + 6$

   (c) $f(x) = 2x^3 - 4x^2 + 8$

   (d) $f(x) = x^2 - 8x + 7$

   (e) $f(x) = 6x^2 + 4x - 6$

Section 3 Further Sketching

We now have a comprehensive range of tools that help us sketch curves, especially those of polynomial functions. We can find intercepts, examine what happens to the function at certain values of x, find critical points, and find properties of critical points. Let's put these tools to use.

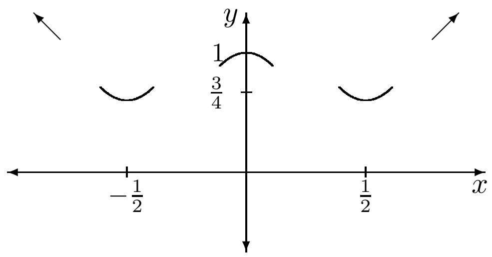
Example 1 : From the information in example 4 in section 2, sketch the function $y = f(x) = x^4 - x^2 + 1$. The critical points were at $x = 0$, and at $x = \pm\frac{1}{\sqrt{2}}$.

At $x = 0$, $f(0) = 1$, and this is a local maximum.

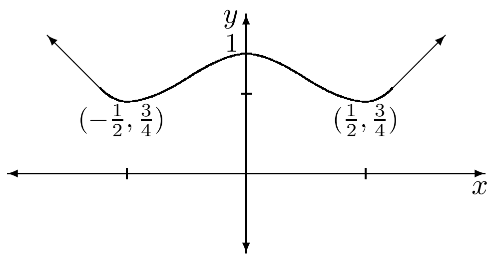
At $x = \frac{1}{\sqrt{2}}$, $f(x) = \frac{3}{4}$ and this point was a local minima.

At $x = -\frac{1}{\sqrt{2}}$, $f(x) = \frac{3}{4}$ and this point was a local minima.

The y-intercept is $(0, 1)$. We leave the question of the x-intercepts for the moment. As $x \to \pm\infty$, $f(x) \to \infty$. From all this information, we can now draw some of $f(x) = x^4 - x^2 + 1$:



Since the three critical points shown are all the critical points, there can be no other changes in direction, so the graph can be completed as follows:



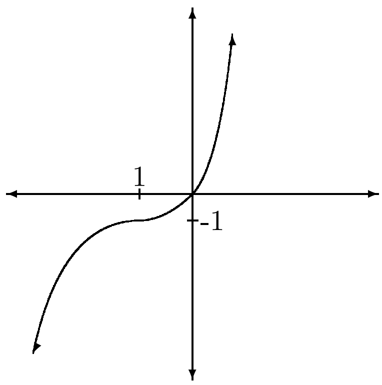
Example 2 : In section 2, example 3, we looked at the function $f(x) = x^3 + 3x^2 + 3x$. Sketch this function. There was only one critical point, $(-1, -1)$, and it was a point of inflection. The y-intercept is found by letting $x = 0$, which gives $f(0) = 0$. The x-axis intercepts are given by the solutions to $f(x) = 0$:

$$x^3 + 3x^2 + 3x = x(x^2 + 3x + 3) = 0$$

So one solution is $x = 0$. The solutions to $x^2 + 3x + 3 = 0$ are given by

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{9 - 12}}{2} \\ &= \frac{-3 \pm \sqrt{-3}}{2} \end{aligned}$$

which has no real solutions. So $x = 0$ is the only x-intercept. As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$. The graph of $f(x) = x^3 + 3x^2 + 3x$ then looks like:



Example 3 : Sketch the function $f(x) = x^3 - 12x + 3$.
$f'(x) = 3x^2 - 12$ which is zero when either $x = 0$ or $x = \pm 2$. $f''(x) = 6x$. We have

$f''(0) = 0$.
$f''(2) = 12 \Rightarrow$ minimum at $(2, -13)$.
$f''(0) = 0 \Rightarrow$ maximum at $(-2, 19)$.

There is an inflection point at $(0, 3)$. The graph of $f(x)$ is shown:

Exercises:

1. Using the method outlined in section 3, sketch the following curves.

   (a) $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 2$

   (b) $f(x) = -x^2 - 2x + 8$

   (c) $f(x) = x^3 - 28x + 48$

   (d) $f(x) = \frac{1}{3}x^3 + 6x^2 + 35$

   (e) $f(x) = x^3 + 6x^2 + 7$

Exercises 3.9 Further Differentiation

1. For each of the following quadratic functions, find the sign of the discriminant to determine if there are 0,1, or 2 roots.

   (a) $f(x) = x^2 - 5x + 6$

   (b) $f(x) = 4x^2 - x + 2$

   (c) $f(x) = 4x^2 - x - 2$

   (d) $f(x) = 4x^2 - 12x + 9$

   (e) $f(x) = 3x^2 - 5x + 3$

2. For each of the following, sketch a function which satisfies the given conditions.

   (a) $f'(x) > 0$ and $f''(x) > 0$

   (b) $f'(x) > 0$ and $f''(x) < 0$

   (c) $f'(x) < 0$ and $f''(x) > 0$

   (d) $f'(x) < 0$ and $f''(x) < 0$

   (e) $f(2) = 4$, $f'(2) = 0$, $f''(x) < 0$ for $x < 2$, and $f''(x) > 0$ for $x > 2$.

3. (a) Sketch the function $f(x) = x^3 - 3x^2 - 9x + 27$, labelling all intercepts and critical points. Determine the nature of the critical points.

   (b) An economist stated: 'Although our current account deficit is increasing, the government policies to reduce it seem to be taking effect'. If D represents the current account deficit, and t represents time, what can be determined about $\frac{dD}{dt}$ and $\frac{d^2D}{dt^2}$?

Answers 3.9

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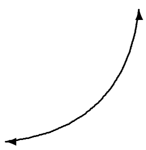
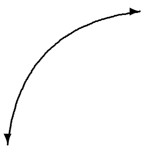
1. (a) $\Delta > 0$, 2 roots
   (b) $\Delta < 0$, no roots
   (c) $\Delta > 0$, 2 roots
   (d) $\Delta = 0$, 1 root
   (e) $\Delta < 0$, no roots

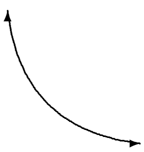
2. (a) Concave up and increasing.



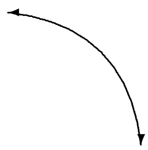
   (b) Concave down and increasing.



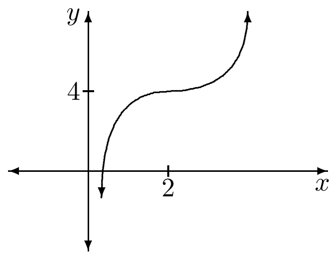
   (c) Concave up and decreasing.



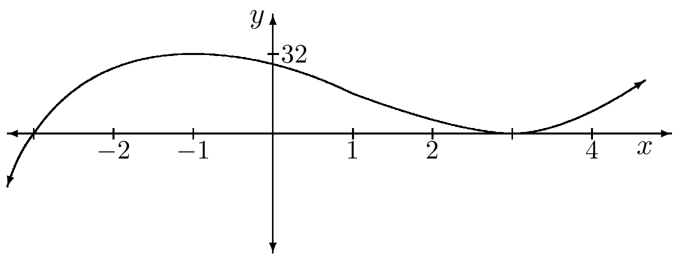
   (d) Concave down and decreasing.



   (e) Change of concavity at $(2, 4)$.



3. (a) $f(x) = (x-3)^2(x+3)$. Intercepts at $x = 3, -3$. $f'(x) = 3(x-3)(x+1)$. Stationary points at $x = 3, -1$. $f''(x) = 6x - 6$. $f''(3) > 0$ so there is a minimum point at $(3, 0)$. $f''(-1) < 0$ so there is a maximum point at $(-1, 32)$.



   (b) $\frac{dD}{dt} > 0$ and $\frac{d^2D}{dt^2} < 0$