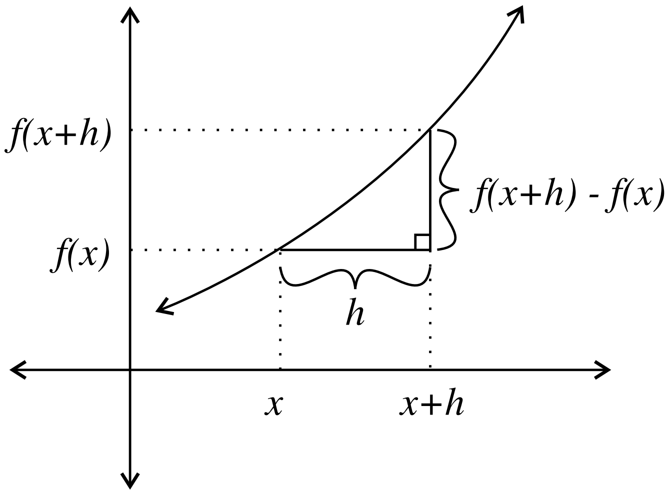
Worksheet 3.8 Introduction to Differentiation

Section 1 Definition of Differentiation

Differentiation is a process of looking at the way a function changes from one point to another. Given any function we may need to find out what it looks like when graphed. Differentiation tells us about the slope (or rise over run, or gradient, depending on the tendencies of your favourite teacher). As an introduction to differentiation we will first look at how the derivative of a function is found and see the connection between the derivative and the slope of the function.



Given the function $f(x)$, we are interested in finding an approximation of the slope of the function at a particular value of x. If we take two points on the graph of the function which are very close to each other and calculate the slope of the line joining them we will be approximating the slope of $f(x)$ between the two points. Our x-values are x and $x+h$, where h is some small number. The y-values corresponding to x and $x+h$ are $f(x)$ and $f(x+h)$. The slope m of the line between the two points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are the two points. In our case, we have the two points $(x, f(x))$ and $(x+h, f(x+h))$. So the slope of the line joining them is given by

$$\begin{aligned} m &= \frac{f(x+h) - f(x)}{x+h-x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

Example 1 : Let $f(x) = x^3$. Find the slope of the line joining $(x, f(x))$ and $(x+h, f(x+h))$. From our definitions,

$$\begin{aligned} m &= \frac{f(x+h) - f(x)}{h} \\ &= \frac{(x+h)^3 - x^3}{h} \\ &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= 3x^2 + 3xh + h^2 \end{aligned}$$

Example 2 : Let $f(x) = 2x + 5$. Find the slope of the line joining the points $(1, f(1))$ and $(1.01, f(1.01))$.

$$\begin{aligned} m &= \frac{f(1.01) - f(1)}{1.01 - 1} \\ &= \frac{7.02 - 7}{0.01} \\ &= \frac{0.02}{0.01} \\ &= 2 \end{aligned}$$

as expected since the gradient of $y = 2x + 5$ is 2.

Example 3 : Let $f(x) = x^2$. Find the slope of the line joining $(x, f(x))$ and $(x+h, f(x+h))$ if $h = 0.1$ and $x = 1$.

$$\begin{aligned} m &= \frac{f(x+h) - f(x)}{h} \\ &= \frac{f(1+0.1) - f(1)}{0.1} \\ &= \frac{f(1.1) - f(1)}{0.1} \\ &= \frac{(1.1)^2 - (1)^2}{0.1} \\ &= \frac{0.21}{0.1} \\ &= 2.1 \end{aligned}$$

The smaller that h gets to zero, the closer x and $x+h$ get to each other, and consequently the better m approximates the slope of the function at the point $(x, f(x))$. So we look at what happens when we take the limit as $h \to 0$ in the slope formula and we call this the derivative $f'(x)$. So

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Notice that $f'(x)$ is the derivative only if the limit exists. If the limit does not exist at particular x-values then we say that the function is not differentiable at those x-values.

Example 4 : Find the derivative of $f(x) = x^2 + 3$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 + 3 - (x^2+3)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 3 - x^2 - 3}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) \\ &= 2x \end{aligned}$$

Note: There are other notations for the derivative of a function in x. The most common are $f'(x)$ and $\frac{df}{dx}$. If $y = f(x)$, we also use $y' = f'(x)$ or $\frac{dy}{dx}$ to refer to the derivative.

Example 5 : Find the derivative of the function $f(x) = 2x + 5$ at $x = 1$.

$$\begin{aligned} f(x+h) &= 2(x+h) + 5 \\ f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{2(1+h) + 5 - 7}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} \\ &= \lim_{h\to 0} 2 \\ &= 2 \end{aligned}$$

Example 6 : Find the derivative of $y = |x|$ at $x = 0$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{|0+h| - |0|}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h} \end{aligned}$$

Recall that

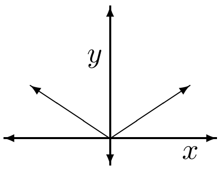
$$|h| = \begin{cases} h & \text{when } h \geq 0 \\ -h & \text{when } h < 0 \end{cases}$$

The absolute-value sign prevents us from simply canceling. Let's look at what $\frac{|h|}{h}$ equals. h could be very small and negative, in which case $f'(0) = -1$. Or it could be very small and positive, in which case $f'(0) = 1$. That is

$$\begin{aligned} &\text{if } h < 0, \text{ then } f'(0) = -1 \\ &\text{if } h > 0, \text{ then } f'(0) = 1 \end{aligned}$$

So the limit does not exist as $h \to 0$ since we get a different value for the limit depending upon whether or not we are close to zero on the negative side or the positive side. Therefore the derivative of $f(x) = |x|$ does not exist at $x = 0$.

Look at the graph of $y = |\,x|$.



The pointed part at $x = 0$ shows a rapid and abrupt change of slope. Functions that have sharp points on their graphs do not have derivatives at these points, although they may have a derivative everywhere else. The function $f(x) = |x|$ is not differentiable at $x = 0$, although it is continuous there.

Exercises:

1. Using the method outlined above, find $f'(x)$ for each of the following functions. That is, use
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
   (a) $f(x) = x^2 + 2$
   (b) $f(x) = 3x - 5$
   (c) $f(x) = 3 - x^2$
   (d) $f(x) = 4x + 5$
   (e) $f(x) = 2 - x$

Section 2 Polynomial Differentiation

Having looked at the general way of finding the derivative of a function, we can now look at those functions for which we already have derivatives and give some simple rules. From these we will be able to determine the derivatives of similar functions. Notice that if we take $f(x) = c$, where c is a constant, we get

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{c - c}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= 0 \end{aligned}$$

The last line is true as $\frac{0}{h} = 0$ for any h except $h = 0$ and limits are about what happens as h gets closer and closer to zero, without actually reaching zero. So for $f(x) = c$ we have $f'(x) = 0$. This is logical since a line such as $y = 2$ which is parallel to the x-axis has a slope of zero.

Now consider $f(x) = ax$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{ax + ah - ax}{h} \\ &= \lim_{h\to 0} \frac{ah}{h} \\ &= a \end{aligned}$$

So if $f(x) = ax$ we get $f'(x) = a$ for any x. Thinking back to worksheet 2.10 where we looked at the function $y = mx + b$, we found that m, the coefficient of x, is the slope of the line. So it makes sense that the derivative of $f(x) = ax$ is a.

Now consider $f(x) = bx^2$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{b(x+h)^2 - bx^2}{h} \\ &= \lim_{h\to 0} \frac{2bxh + bh^2}{h} \\ &= \lim_{h\to 0} (2bx + bh) \\ &= 2bx \end{aligned}$$

So if $f(x) = bx^2$ we get $f'(x) = 2bx$ for any x. We could carry on for higher powers of x and notice the pattern that if

$$\begin{aligned} f(x) &= cx^n \\ \text{then} \quad f'(x) &= ncx^{n-1} \end{aligned}$$

Furthermore, if we have a sum of functions, it can be shown that the derivative of the sum is the sum of the derivatives. This means that if $f(x)$ is a sum of terms that each look like cx^n (in other words, a polynomial) you can use the above rule for each term to determine the derivative of the function.

Example 1 : Let $g(x) = x^2 + 3x + 2$. Then

$$\begin{aligned} g'(x) &= 2x + 3 + 0 \\ &= 2x + 3 \end{aligned}$$

Example 2 : Find the derivative of $f(x) = 5x^3 + 3x^2 + 2^2$.

$$\begin{aligned} f'(x) &= 5 \times 3x^2 + 3 \times 2x^1 + 0 \\ &= 15x^2 + 6x \end{aligned}$$

Example 3 : Find the derivative of $h(x) = 1 + \frac{1}{x} = 1 + x^{-1}$.

$$\begin{aligned} h'(x) &= 0 + -1 \times x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

Example 4 : Given $f(x) = 6x^2 - 4x + 1$, find $f'(2)$.
First find $f'(x)$, then find $f'(2)$.

$$\begin{aligned} f'(x) &= 12x - 4 \\ f'(2) &= 12 \times 2 - 4 \\ &= 20 \end{aligned}$$

Example 5 : Given $f(x) = x^3 - 2x^2 + 5$, find $f'(-1)$.
First find $f'(x)$, then find $f'(-1)$.

$$\begin{aligned} f'(x) &= 3x^2 - 4x \\ f'(-1) &= 3(-1)^2 - 4(-1) \\ &= 7 \end{aligned}$$

Exercises:

1. Find the derivative of each of the following functions

   (a) x^2
   (b) $3x^2 + 4x$
   (c) $x^3 - 6x$
   (d) $6x^2 - 2x + 3$
   (e) $\frac{1}{x} + 4x$
   (f) $3x^2 - x + 2$
   (g) $x^5 + 4x^3 - 7x$
   (h) $\frac{4}{x} - x^2$
   (i) $4x^5 + 6x^3$
   (j) $x^7 + 4x^5$

2. (a) If $f(x) = 2x^3 - 4x$, find $f'(3)$.
   (b) If $f(x) = 7x^2 - 2$, find $f'(-4)$.
   (c) If $f(x) = 5 - 3x^2$, find $f'(1)$.
   (d) If $f(x) = 6x + 7$, find $f'(12)$.
   (e) If $f(x) = 4x^3 - 2x^2 + 4x$, find $f'(5)$.

Section 3 Stationary points

Recall that differentiation tells us about the slope of a function at any point on the graph where the function is defined. If $f'(5) = 3$ then, for the function f, we know that the slope of the function at $x = 5$ is 3. If we know the equation of a function, then we could evaluate the slope at various x-values. There are particular points on a graph which are of special interest. These are called stationary points. At a stationary point, the gradient of the function is zero. The stationary points are of interest to us because they help us to draw the graph of a function. There are three different types of stationary points:

1. Minimum points
2. Maximum points
3. Horizontal points of inflection

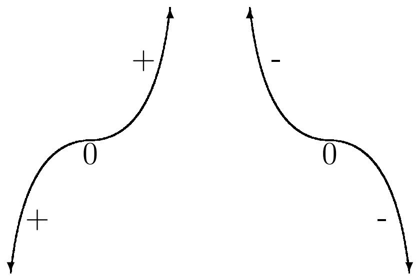
All types of stationary points have the property that the derivative is zero.

Minimum points occur when the graph reaches a local minimum, and has a shape like this:

The gradient changes from negative to zero to positive. We call this concave up, because the cup opens upwards. A maximum occurs when the graph looks like this:

We call this concave down. The gradient changes from positive to zero to negative.

The third type of stationary point, a horizontal point of inflection, occurs when the concavity changes from up to down or from down to up. They look like:



The slope of a horizontal point of inflection momentarily goes to zero where the curve changes concavity. On either side of the point of inflection, the gradient has the same sign, i.e. if the gradient is negative on one side of the point of inflection, then it is negative on the other side. Conversely, for a stationary point that is either a minimum or a maximum, the gradient is negative on one side of the point and positive on the other.

Using this information, we can determine what types of stationary points occur on a graph.

Example 1 : Find the slope of the function $f(x) = x^3 + 3$ at $x = 1$ and $x = 0$.

$$\begin{aligned} f'(x) &= 3x^2 + 0 \\ &= 3x^2 \end{aligned}$$

The slope of the function f at $x = 1$ is the value of the derivative at $x = 1$. So $f'(1) = 3(1)^2 = 3$. The slope of the function $f(x) = x^3 + 3$ at $x = 1$ is 3.

The slope at $x = 0$ is found in the same way. $f'(0) = 3(0)^2 = 0$. The slope at $x = 0$ is 0 so there must be a stationary point at $x = 0$.

Example 2 : Find all the stationary points of $g(x) = x^2 + 2x + 2$.

We have $g'(x) = 2x + 2$. Stationary points occur when $g' = 0$. So we must find all x for which this is true, i.e. we need to solve the equation

$$2x + 2 = 0$$

The only solution for this is $x = -1$, so $x = -1$ is the x-value of the stationary point. To find the other coordinate, we put $x = -1$ in the original function:

$$g(-1) = (-1)^2 + 2(-1) + 2 = 1$$

The only stationary point is $(-1, 1)$. To see what kind of stationary point it is we need to see what the slope is on either side of the stationary point. Now, $-1 + h$ is on the right side of -1 for h small and positive, and on the left side of -1 for h small and negative. The slope at $-1 + h$ is

$$\begin{aligned} g'(-1+h) &= 2(-1+h) + 2 \\ &= -2 + 2h + 2 \\ &= 2h \end{aligned}$$

This is positive for h positive, and negative for h negative. This means that the stationary point is a local minimum.

Example 3 : Find the stationary points of the function $f(x) = x^3 + 3x^2 + 5$.

We calculate $f'(x)$ and find all the x-values that satisfy $f'(x) = 0$.

$$f'(x) = 3x^2 + 6x = 3x(x + 2) = 0$$

This equation has the solutions $x = 0$ and $x = -2$. And $f(0) = 5$, $f(-2) = 9$. So there are two stationary points: $(0, 5)$ and $(-2, 9)$.

Remember that the derivative at any value of x gives you the slope of the function at that value of x.

Example 4 : Given $f(x) = 3x^2 + 2x + 1$, find $f'(-3)$.

First find $f'(x)$, then find $f'(-3)$.

$$\begin{aligned} f'(x) &= 6x + 2 \\ f'(-1) &= 6(-3) + 2) \\ &= -16 \end{aligned}$$

Example 5 : Given $f(x) = 4x^2 - 5x + 7$, find the value of x for which $f'(x) = 11$.

First find $f'(x)$, then solve $f'(x) = 11$.

$$\begin{aligned} f'(x) &= 8x - 5 \\ 11 &= 8x - 5 \\ x &= 2 \end{aligned}$$

When $x = 2$ the slope of the function is 11.

Exercises:

1. Given $f(x) = -2x^2 + 6x - 4$
   (a) find x for which $f'(x) = 20$
   (b) find $f'(2)$
2. Find the stationary points for each of the following functions and state whether they are a maximum, minimum, or a point of inflection.
   (a) $f(x) = x^2 + 6x + 8$
   (b) $f(x) = -x^2 - 2x + 15$
   (c) $f(x) = x^3 + 2$

Section 4 Sketching Graphs

We can use information that we get from derivatives to help sketch graphs of functions. If we can determine the x- and y-intercepts of a function together with the stationary points we can determine roughly what the function looks like. The other bit of information we can find useful is what happens to the function as x approaches positive or negative infinity.

Example 1 : Use the derivatives to help sketch $f(x) = x^3 + 3x^2 + 5$.

As determined in the previous section, $f'(x) = 3x^2 + 6x$, and f has the two stationary points $(0, 5)$ and $(-2, 9)$. Now we should determine the changes of slope on either side of both these points. Look at the x-value $0 + h$, which is to the right of 0 if h is small and positive, and to the left of 0 if h is small and negative. We have $f'(0+h) = f'(h) = 3(h)^2 + 6(h) = 3h^2 + 6h$. This is positive if h is small and positive, and negative if h is small and negative. The slope is positive to the right and negative to the left. Then $(0, 5)$ is a minimum turning point.

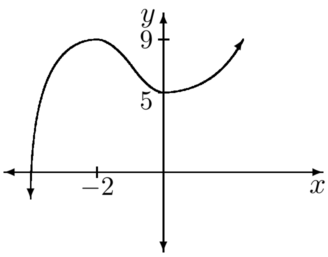
Repeating the process for $x = -2$, we get

$$\begin{aligned} f'(-2+h) &= 3(-2+h)^2 + 6(-2+h) \\ &= -6h + 3h^2 \\ &= 3h(h-2) \end{aligned}$$

This is negative if h is small and positive, but positive if h is small and negative. The slope is positive to the left of -2 , and negative to the right of -2. Then $(-2, 9)$ is a local maximum.

In addition, as $x \to \infty$, $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to -\infty$.

So the graph looks like:



Example 2 : Sketch the graph of $f(x) = x^3 + 3$.

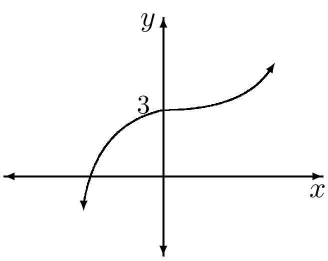
We first solve

$$f'(x) = 3x^2 = 0$$

This has the solution $x = 0$. So the critical point is $(0, 3)$. What happens either side of $x = 0$?

$$f'(0+h) = 3h^2$$

This is positive for positive or negative h, so we have a horizontal point of inflection. The slope to the left and right of $x = 0$ is positive. The graph looks like:



Exercises:

1. Sketch the following graphs using the process outlined in section 4.

   (a) $f(x) = x^2 + 4x + 3$

   (b) $f(x) = x^3 - 1$

   (c) $f(x) = 2x^3 - 3x^2 - 36x + 18$

   (d) $f(x) = -x^2 - 2x + 15$

   (e) $f(x) = x^2 - 2x - 24$

Exercises 3.8 Introduction to Differentiation

1. Using the definition
$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
evaluate the derivative of the functions given.

   (a) $f(x) = 2x + 3$

   (b) $f(x) = x^2 - 2x + 1$

   (c) $f(x) = x^3$

2. Using the rule $\frac{dx^n}{dx} = nx^{n-1}$, differentiate the following functions with respect to x.

   (a) $x^2 + 6x + 83$

   (b) $7x^3 - 5x^2 + 9x$

   (c) $\sqrt{x} + 8x$

   (d) $3x^{-2} + x^{-1}$

   (e) $\frac{1}{x^2} + \frac{1}{x} + 6x$ (Hint: rewrite $\frac{1}{x^2}$ as x^{-2}.)

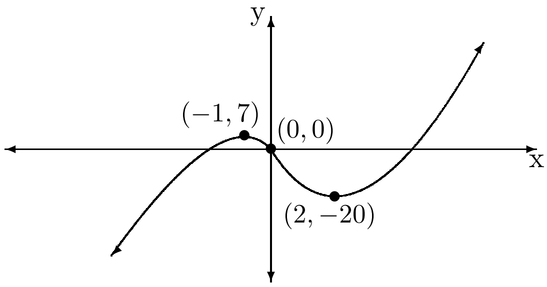
3. (a) Sketch the function $f(x) = 2x^3 - 3x^2 - 12x$, labelling the y-intercepts and the stationary points.

   (b) The air temperature T (degrees Celsius) as a function of height s (kilometres) above sea level is measured by a scientist in a hot-air balloon. The function is given by $T = 20 - 3s$. Find $T'(s)$ and give an interpretation of your answer.

Answers 3.8

2. (a) $2x + 6$
   (b) $21x^2 - 10x + 9$
   (c) $\frac{1}{2\sqrt{x}} + 8$
   (d) $-6x^{-3} - x^{-2}$
   (e) $-\frac{2}{x^3} - \frac{1}{x^2} + 6$

3. (a)



   (b) $T'(s) = -3$. The temperature is dropping 3 degrees for every km above sea level.