Worksheet 3.7 Continuity and Limits

Section 1 Limits

Limits were mentioned without very much explanation in the previous worksheet. We will now take a closer look at limits and, in particular, the limits of functions. Limits are very important in maths, but more specifically in calculus.

To begin with, we will look at two geometric progressions:

1. $2, 4, 8, 16, \ldots$
2. $5, \frac{1}{2}, \frac{1}{20}, \frac{1}{200}, \ldots$

In the first geometric progression, successive terms get larger and larger as we go along the list. Recall from the last worksheet that the nth term for this geometric progression is

$$u_n = 2 \times 2^{n-1}$$

As n increases, u_n gets larger and larger, and we can make u_n as large as we wish by taking a suitable value for n. The second geometric progression also has infinitely many terms, but in this case the terms are getting smaller and smaller as the list goes on. The nth term for this geometric progression is

$$u_n = 5(\frac{1}{10})^{n-1}$$

So u_{10}, say, is

$$\begin{aligned} u_{10} &= 5(\frac{1}{10})^{10-1} \\ &= \frac{5}{10^9} \\ &= 0.000000005 \end{aligned}$$

which is very small. Indeed, as we take n larger and larger, the terms seem to be getting nearer to zero. We can make the nth term as close as we like to zero by taking a suitably large n. Note that, even though the terms are getting nearer to zero, they will never actually equal zero, no matter how large we make n. But what we do say is that the limit of the geometric progression is zero and we write this as

$$\lim_{n\to\infty} \frac{5}{10^{n-1}} = 0$$

We read this statement as follows: the limit as n tends to ∞ of $\frac{5}{10^{n-1}}$ is zero. In the first geometric progression that we looked at, where the terms got bigger and bigger as n increased, we say that that geometric progression has no limit.

We can find the limit of a function $f(x)$ as $x \to \infty$. For a given function, we will look at what happens as x takes on larger and larger values and work out a general trend. Let's look at the function

$$f(x) = \frac{1}{x}$$

For large values of x, $f(x)$ is very small. As x gets larger, $f(x) \to 0$, so we can say that

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

even though there is no x such that $f(x) = 0$. Now we investigate $f(x) = \frac{1}{x}$ as $x \to 0$. So we are trying to find

$$\lim_{x\to 0} \frac{1}{x}$$

We takes values of x closer and closer to zero and see what happens.

$$\begin{aligned} f(1) &= 1 \\ f(0.01) &= 100 \\ f(0.00001) &= 100000 \end{aligned}$$

It appears that $f(x)$ is getting bigger and bigger as $x \to 0$. Therefore

$$\lim_{x\to 0} \frac{1}{x} \quad \text{does not exist}$$

Notice that $f(x) = \frac{1}{x}$ is not actually defined at $x = 0$ as we are not allowed to divide by zero. Instead of approaching zero by starting at 1 and getting smaller, what happens if we start at $x = -1$ and approach zero from there?

$$\begin{aligned} f(-1) &= -1 \\ f(-0.01) &= -100 \\ f(-0.00001) &= -100,000 \end{aligned}$$

Again, the limit does not exist, but now $f(x)$ gets further away from zero in the negative direction as x gets closer to zero. When looking at the limit of a function as it tends to some finite value, it is important to check values of x on both sides of the value you are looking at.

Example 1 : If $f(x) = 1 - 2x$ find

$$\lim_{x\to 0} f(x)$$

We try a few values:

$$\begin{aligned} f(0.01) &= .98 \\ f(-0.01) &= 1.02 \\ f(0.0001) &= .9998 \\ f(-0.0001) &= 1.0002 \end{aligned}$$

These seem to be getting closer to 1 as $x \to 0$, and if we evaluate the function at $x = 0$ we get $f(0) = 1$. This is a convenient check, but remember that limits are actually looking at what happens to a function as x approaches a certain point.

Example 2 : Evaluate $\lim_{x\to 3}(5x + 2)$. It is easy to see that as $x \to 3$ (from both directions) that $f(x)$ approaches 17.

Example 3 : Evaluate

$$\lim_{x\to 5} \frac{x^2 - 25}{x - 5}$$

The function as written is not defined at $x = 5$ since putting $x = 5$ would give us a zero denominator, but we can factorize the numerator to give

$$\begin{aligned} \lim_{x\to 5} \frac{x^2 - 25}{x - 5} &= \lim_{x\to 5} \frac{(x+5)(x-5)}{x-5} \\ &= \lim_{x\to 5}(x + 5) \\ &= 10 \end{aligned}$$

The step where we divide the numerator and denominator by $x - 5$ is only valid for $x \neq 5$, but the point of limits is to look at what is happening close to 5, not actually at 5.

Example 4 :

$$\begin{aligned} \lim_{x\to 0} \frac{x^3 + 2x^2}{x^2} &= \lim_{x\to 0} \frac{x^2(x+2)}{x^2} \\ &= \lim_{x\to 0} x + 2 \\ &= 2 \end{aligned}$$

Sometimes we are asked to find the limit as $x \to \infty$ of a function, and it is unclear what the limit is. For instance, in the case of

$$f(x) = \frac{3x^2 + x + 2}{2x^2 + 3x + 1}$$

we can't cancel any factors or simplify the expression. Since both the numerator and denominator get large as x gets large, it is not clear whether $f(x)$ gets large or not. We now discuss two methods that we can use to find the limit in cases like this.

Method A

We can divide the numerator and the denominator by the highest power of x in the denominator.

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{3x^2+x+2}{2x^2+3x+1} \\ &= \lim_{x\to\infty} \frac{3+\frac{1}{x}+\frac{2}{x^2}}{2+\frac{3}{x}+\frac{1}{x^2}} \\ &= \frac{3}{2} \end{aligned}$$

The first step is to divide every term in both the numerator and the denominator by x^2. The second, and last, step follows because all terms except the 3 on top and the 2 on the bottom approach 0 as x approaches ∞.

Method B

Recall that as $x \to \infty$ then $\frac{1}{x} \to 0$, and as $x \to 0$ then $\frac{1}{x} \to \infty$. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0} f(\frac{1}{x})$$

To find the limit as $x \to \infty$ of $f(x)$, we can equivalently look at $x \to 0$ of $f(\frac{1}{x})$.

$$\begin{aligned} \lim_{x\to\infty} \frac{3x^2+x+2}{2x^2+3x+1} &= \lim_{x\to 0} \frac{3\frac{1}{x^2}+\frac{1}{x}+2}{2\frac{1}{x^2}+3\frac{1}{x}+1} \\ &= \lim_{x\to 0} \frac{\frac{3+x+2x^2}{x^2}}{\frac{2+3x+x^2}{x^2}} \\ &= \frac{3}{2} \end{aligned}$$

Exercises:

1. Find the following limits, if they exist.

   (a) $\lim_{x\to 3}(4x+1)$

   (b) $\lim_{x\to 2} \frac{x^2-4}{x-2}$

   (c) $\lim_{x\to\infty} \frac{4x^2-2x+7}{3x^2+6x-5}$

   (d) $\lim_{x\to 0} \frac{6}{x}$

   (e) $\lim_{x\to -3} \frac{x^2+5x+6}{2x+6}$

   (f) $\lim_{x\to 1} x^2+3$

(g) $\lim_{x \to 0} \dfrac{x+2}{x^2}$

(h) $\lim_{x \to 6} \dfrac{3x-18}{x^2-36}$

(i) $\lim_{x \to \infty} 2x+1$

(j) $\lim_{x \to \infty} \dfrac{1}{x} - 4$

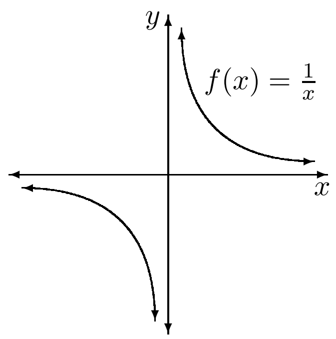
Section 2 Continuity

Limits help to sketch the graphs of functions on the $x-y$ plane. They tell how the function behaves as it gets close to certain values of x and what value the function tends to as x gets large, both positively and negatively.

If the limit of a function does not exist at a certain finite value of x, then the function is discontinuous at that point.

Example 1 : Given that $f(x) = \frac{1}{x}$, we know that $\lim_{x\to\infty} f(x) = 0$ and that $\lim_{x\to 0} f(x)$ does not exist. The function $f(x)$ is not defined at $x = 0$.

The graph of $y = f(x)$ is drawn below:



For a function to be continuous at $x = c$ we need three conditions to be met.

1. $f(x)$ must be defined at $x = c$
2. $\lim_{x\to c} f(x)$ must exist
3. $f(c)$ must equal $\lim_{x\to c} f(x)$

Note that $f(x) = \frac{1}{x}$ is not continuous at $x = 0$ because $f(0)$ is not defined; neither does $\lim_{x\to 0} f(x)$ exist.

When asked to test for continuity, the first thing that we check for is whether or not the function is defined at the point in question. Then we can check the limit of the function as x tends to that value. Finally, we would check that $f(c) = \lim_{x\to c} f(x)$.

Example 2 : We define $f(x)$ as

$$f(x) = \begin{cases} \frac{x^2+9}{2} & \text{when } x \neq 0 \\ 5 & \text{when } x = 0 \end{cases}$$

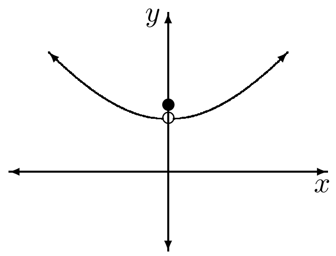
Is the function continuous at $x = 0$? The function is defined at zero: $f(0) = 5$.

$$\lim_{x\to 0} \frac{x^2+9}{2} = 4\frac{1}{2}$$

Notice that we are looking at values close to zero, but not actually zero, so we use the appropriate part of the function. The limit exists as $x \to 0$. But

$$\lim_{x\to 0} \frac{x^2+9}{2} = 4\frac{1}{2} \neq f(0)$$

Therefore the function is not continuous at $x = 0$. The graph of $y = f(x)$ is drawn below:



In general, for simple functions, there is a rule of thumb that says that if you can draw the graph of the function without lifting your pen from the paper, then the function is almost certainly continuous for those values of x.

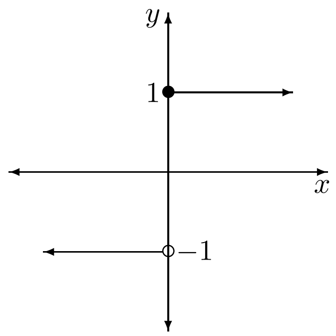
Example 3 : The function $f(x) = \frac{3}{x+1}$ is not continuous at $x = -1$ since it is not defined at $x = -1$.

Example 4 : The function $f(x) = \frac{5x}{x^2+4x+3} = \frac{5x}{(x+3)(x+1)}$ is not continuous at $x = -1$ and $x = -3$ since it is not defined at those values.

Example 5 : The function

$$f(x) = \begin{cases} -1 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases}$$

is defined for all values of x, but $\lim_{x\to 0} f(x)$ does not exist, since if we take values of x close to zero on the negative side we get -1, but if we take values of x close to zero on the positive side we get +1. Therefore $f(x)$ is discontinuous at $x = 0$.



Exercises:

1. Which of the following limits exist? If they exist, evaluate them.

   (a) $\lim\limits_{x\to 2} \dfrac{3}{x-2}$

   (b) $\lim\limits_{x\to 4} \dfrac{x+4}{2}$

   (c) $\lim\limits_{x\to 0} \dfrac{x^2+3x}{2x}$

   (d) $\lim\limits_{x\to \infty} \dfrac{x^2+6x+8}{5x^2+4}$

   (e) $\lim\limits_{x\to 0} \dfrac{6}{x}$

Exercises 3.7 Continuity and Limits

1. Evaluate the following:

   (a) $\lim_{x\to 3}(2x+4)$

   (b) $\lim_{x\to 3}\frac{x^2-9}{x-3}$

   (c) $\lim_{x\to 3}\frac{1}{x}$

   (d) $\lim_{x\to 3}\frac{3x^2-2x+4}{4x^2-7}$

   (e) $\lim_{x\to 3}\frac{(2x-6)(x+3)}{\sqrt{x^2-9}}$

2. Which of the following are continuous at $x=1$?

   (a) $f(x)=\begin{cases} 5 & x=1 \\ 2x+3 & x\neq 1 \end{cases}$

   (b) $f(x)=\begin{cases} 4x & x=1 \\ \frac{2x^2-2}{x-1} & x\neq 1 \end{cases}$

   (c) $f(x)=\begin{cases} 0 & x=1 \\ \frac{x-1}{\sqrt{1-x}} & x\neq 1 \end{cases}$

   (d) $f(x)=\begin{cases} \left|2x-3\right| & x=1 \\ 2x-3 & x\neq 1 \end{cases}$

   (e) $f(x)=\begin{cases} x-3 & x\leq 1 \\ x^2-4x+3 & x>1 \end{cases}$

Answers 3.7

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1. (a) 10
   (b) 6
   (c) $\frac{1}{3}$
   (d) $\frac{25}{29}$
   (e) 0

2. (a) Continuous
   (b) Continuous
   (c) Continuous
   (d) Not continuous
   (e) Not continuous