Worksheet 3.6 Arithmetic and Geometric Progressions

Section 1 Arithmetic Progression

An arithmetic progression is a list of numbers where the difference between successive numbers is constant. The terms in an arithmetic progression are usually denoted as u_1, u_2, u_3 etc. where u_1 is the initial term in the progression, u_2 is the second term, and so on; u_n is the nth term. An example of an arithmetic progression is

$$2, 4, 6, 8, 10, 12, 14, \ldots$$

Since the difference between successive terms is constant, we have

$$u_3 - u_2 = u_2 - u_1$$

and in general

$$u_{n+1} - u_n = u_2 - u_1$$

We will denote the difference $u_2 - u_1$ as d, which is a common notation.

> Example 1 : Given that 3,7 and 11 are the first three terms in an arithmetic progression, what is d?
>
> $$7 - 3 = 11 - 7 = 4$$
>
> Then $d = 4$. That is, the common difference between the terms is 4.

If we know the first term in an arithmetic progression , and the difference between terms, then we can work out the nth term, i.e. we can work out what any term will be. The formula which tells us what the nth term in an arithmetic progression is

$$u_n = a + (n - 1) \times d$$

where a is the first term.

> Example 2 : If the first 3 terms in an arithmetic progression are 3,7,11 then what is the 10th term? The first term is $a = 3$, and the common difference is $d = 4$.
>
> $$\begin{aligned} u_n &= a + (n-1)d \\ u_{10} &= 3 + (10 - 1)4 \\ &= 3 + 9 \times 4 \\ &= 39 \end{aligned}$$

Example 3 : If the first 3 terms in an arithmetic progression are 8,5,2 then what is the 16th term? In this progression $a = 8$ and $d = -3$.

$$\begin{aligned} u_n &= a + (n-1)d \\ u_{16} &= 8 + (10-1) \times (-3) \\ &= -37 \end{aligned}$$

Example 4 : Given that $2x, 5$ and $6 - x$ are the first three terms in an arithmetic progression , what is d?

$$\begin{aligned} 5 - 2x &= (6 - x) - 5 \\ x &= 4 \end{aligned}$$

Since $x = 4$, the terms are 8, 5, 2 and the difference is -3. The next term in the arithmetic progression will be -1.

An arithmetic series is an arithmetic progression with plus signs between the terms instead of commas. We can find the sum of the first n terms, which we will denote by S_n, using another formula:

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Example 5 : If the first 3 terms in an arithmetic progression are 3,7,11 then what is the sum of the first 10 terms?
Note that $a = 3$, $d = 4$ and $n = 10$.

$$\begin{aligned} S_{10} &= \frac{10}{2}(2 \times 3 + (10-1) \times 4) \\ &= 5(6 + 36) \\ &= 210 \end{aligned}$$

Alternatively, but more tediously, we add the first 10 terms together:

$$S_{10} = 3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39 = 210$$

This method would have drawbacks if we had to add 100 terms together!

Example 6 : If the first 3 terms in an arithmetic progression are 8,5,2 then what is the sum of the first 16 terms?

$$\begin{aligned} S_{16} &= \frac{16}{2}(2 \times 8 + (16-1) \times (-3)) \\ &= 8(16 - 45) \\ &= -232 \end{aligned}$$

Exercises:

1. For each of the following arithmetic progressions, find the values of a, d, and the u_n indicated.

   (a) 1, 4, 7, ..., (u_{10})
   (b) −8, −6, −4, ..., (u_{12})
   (c) 8, 4, 0, ..., (u_{20})
   (d) −20, −15, −10, ..., (u_6)
   (e) 40, 30, 20, ..., (u_{18})
   (f) −6, −8, −10, ..., (u_{12})
   (g) 2, $2\frac{1}{2}$, 3, ..., (u_{19})
   (h) 6, $5\frac{3}{4}$, $5\frac{1}{2}$, ..., (u_{10})
   (i) −7, $-6\frac{1}{2}$, −6, ..., (u_{14})
   (j) 0, −5, −10, ..., (u_{15})

2. For each of the following arithmetic progressions, find the values of a, d, and the S_n indicated.

   (a) 1, 3, 5, ..., (S_8)
   (b) 2, 5, 8, ..., (S_{10})
   (c) 10, 7, 4, ..., (S_{20})
   (d) 6, $6\frac{1}{2}$, 7, ..., (S_8)
   (e) −8, −7, −6, ..., (S_{14})
   (f) −2, 0, 2, ..., (S_5)
   (g) −20, −16, −12, ..., (S_4)
   (h) 40, 35, 30, ..., (S_{11})
   (i) 12, $10\frac{1}{2}$, 9, ..., (S_9)
   (j) −8, −5, −2, ..., (S_{20})

Section 2 Geometric Progressions

A geometric progression is a list of terms as in an arithmetic progression but in this case the ratio of successive terms is a constant. In other words, each term is a constant times the term that immediately precedes it. Let's write the terms in a geometric progression as u_1, u_2, u_3, u_4 and so on. An example of a geometric progression is

$$10, 100, 1000, 10000, \ldots$$

Since the ratio of successive terms is constant, we have

$$\begin{aligned} \frac{u_3}{u_2} &= \frac{u_2}{u_1} \quad \text{and} \\ \frac{u_{n+1}}{u_n} &= \frac{u_2}{u_1} \end{aligned}$$

The ratio of successive terms is usually denoted by r and the first term again is usually written a.

Example 1 : Find r for the geometric progression whose first three terms are 2, 4, 8.

$$\frac{4}{2} = \frac{8}{4} = 2$$

Then $r = 2$.

Example 2 : Find r for the geometric progression whose first three terms are 5, $\frac{1}{2}$, and $\frac{1}{20}$.

$$\frac{1}{2} \div 5 = \frac{1}{20} \div \frac{1}{2} = \frac{1}{10}$$

Then $r = \frac{1}{10}$.

If we know the first term in a geometric progression and the ratio between successive terms, then we can work out the value of any term in the geometric progression . The nth term is given by

$$\boxed{u_n = ar^{n-1}}$$

Again, a is the first term and r is the ratio. Remember that $ar^{n-1} \neq (ar)^{n-1}$.

Example 3 : Given the first two terms in a geometric progression as 2 and 4, what is the 10th term?

$$a = 2 \qquad r = \frac{4}{2} = 2$$

Then $u_{10} = 2 \times 2^9 = 1024$.

Example 4 : Given the first two terms in a geometric progression as 5 and $\frac{1}{2}$, what is the 7th term?

$$a = 5 \qquad r = \frac{1}{10}$$

Then

$$\begin{aligned} u_7 &= 5 \times (\frac{1}{10})^{7-1} \\ &= \frac{5}{1000000} \\ &= 0.000005 \end{aligned}$$

A geometric series is a geometric progression with plus signs between the terms instead of commas. So an example of a geometric series is

$$1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

We can take the sum of the first n terms of a geometric series and this is denoted by S_n:

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

Example 5 : Given the first two terms of a geometric progression as 2 and 4, what is the sum of the first 10 terms? We know that $a = 2$ and $r = 2$. Then

$$\begin{aligned} S_{10} &= \frac{2(1-2^{10})}{1-2} \\ &= 2046 \end{aligned}$$

Example 6 : Given the first two terms of a geometric progression as 5 and $\frac{1}{2}$, what is the sum of the first 7 terms? We know that $a = 5$ and $r = \frac{1}{10}$. Then

$$\begin{aligned} S_7 &= \frac{5(1 - \frac{1}{10}^7)}{1 - \frac{1}{10}} \\ &= 5\frac{1 - \frac{1}{10^7}}{\frac{9}{10}} \\ &= 5.555555 \end{aligned}$$

In certain cases, the sum of the terms in a geometric progression has a limit (note that this is summing together an infinite number of terms). A series like this has a limit partly because each successive term we are adding is smaller and smaller (but this fact in itself is not enough to say that the limiting sum exists). When the sum of a geometric series has a limit we say that S_∞ exists and we can find the limit of the sum. For more information on limits, see worksheet 3.7. The condition that S_∞ exists is that r is greater than -1 but less than 1, i.e. $|r| < 1$. If this is the case, then we can use the formula for S_n above and let n grow arbitrarily big so that r^n becomes as close as we like to zero. Then

$$\boxed{S_\infty = \tfrac{a}{1-r}}$$

is the limit of the geometric progression so long as $-1 < r < 1$.

Example 7 : The geometric progression whose first two terms are 2 and 4 does not have a S_∞ because $r = 2 \not< 1$.

Example 8 : For the geometric progression whose first two terms are 5 and $\frac{1}{2}$, find S_∞. Note that $r = \frac{1}{10}$ so $|r| < 1$, so that S_∞ exists. Now

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{5}{1-\frac{1}{10}} \\ &= 5\frac{5}{9} \end{aligned}$$

So the sum of $5 + \frac{1}{2} + \frac{1}{20} + \frac{1}{200} + \ldots$ is $5\frac{5}{9}$

Example 9 : Consider a geometric progression whose first three terms are 12, -6 and 3. Notice that $r = -\frac{1}{2}$. Find both S_8 and S_∞.

$$\begin{aligned} S_8 &= \frac{a(1-r^n)}{1-r} \\ &= \frac{12(1-(-\frac{1}{2})^8)}{1-(-\frac{1}{2})} \\ &\approx 7.967 \end{aligned}$$

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{12}{1-(-\frac{1}{2})} \\ &= \frac{12}{3/2} \\ &= 8 \end{aligned}$$

Exercises:

1. Find the term indicated for each of the geometric progressions.

   (a) 1, 3, 9, . . ., (u_9)

   (b) 4, -8, 16, . . ., (u_{10})

   (c) 18, -6, 2, . . ., (u_{12})

   (d) 1000, 100, 10, . . ., (u_7)

   (e) 32, -8, 2, . . ., (u_{14})

   (f) -0.005, -0.05, -0.5, . . ., (u_{10})

   (g) -6, -12, -24, . . ., (u_6)

   (h) 1.4, 0.7, 0.35, . . ., (u_5)

   (i) 68, -34, 17, . . ., (u_9)

   (j) 8, 2, $\frac{1}{2}$, . . ., (u_{11})

2. Find the sum indicated for each of the following geometric series

   (a) $6 + 9 + 13.5 + \cdots \ (S_{10})$
   (b) $18 - 9 + 4.5 + \cdots \ (S_{12})$
   (c) $6 + 3 + \frac{3}{2} + \cdots \ (S_{10})$
   (d) $6000 + 600 + 60 + \cdots \ (S_{20})$
   (e) $80 - 20 + 5 + \cdots \ (S_9)$

Exercises 3.6 Arithmetic and Geometric Progressions

1. For each of the following progressions, determine whether it is arithmetic, geometric, or neither:

   (a) 5, 9, 13, 17, ...

   (b) 1, -2, 4, -8, ...

   (c) 1, 1, 2, 3, 5, 8, 13, 21, ...

   (d) 81, -9, 3, $\frac{1}{3}$, ...

   (e) 512, 474, 436, 398, ...

2. Find the sixth and twentieth terms, and the sum of the first 10 terms of each of the following sequences:

   (a) -15, -9, -3, ...

   (b) $\log 7$, $\log 14$, $\log 28$, ...

   (c) $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, ...

   (d) 0.5, 0.45, 0.405, ...

   (e) 64,-32,16, ...

3. (a) The third and eighth terms of an AP are 470 and 380 respectively. Find the first term and the common difference. Hint: write expressions for u_3 and u_8 and solve simultaneously.

   (b) Find the sum to 5 terms of the geometric progression whose first term is 54 and fourth term is 2.

   (c) Find the second term of a geometric progression whose third term is $\frac{9}{4}$ and sixth term is $-\frac{16}{81}$.

   (d) Find the sum to n terms of an arithmetic progression whose fourth and fifth terms are 13 and 15.

4. (a) A university lecturer has an annual salary of \$40,000. If this increases by 2% each year, how much will she have grossed in total after 10 years?

   (b) A bob of a pendulum swings through an arc of 50 cm on its first swing. Each successive swing is 90% of the length of the previous swing. Find the total distance the bob travels before coming to rest.

Answers 3.6

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1. (a) Arithmetic (b) Geometric (c) Neither (d) Neither (e) Arithmetic

2. (a) $T_6 = 15$, $T_{20} = 99$, $S_{10} = 120$
   (b) $T_6 = \log 7 + 5\log 2$, $T_{20} = \log 7 + 19\log 2$, $S_{10} = \frac{10}{2}(2\log 7 + 9\log 2)$
   (c) $T_6 = 2$, $T_{20} = 2^{15}$, $S_{10} = \frac{1}{16}(2^{10} - 1)$
   (d) $T_6 = (0.5)(0.9)^5$, $T_{20} = (0.5)(0.9)^{19}$, $S_{10} = 5(1 - .9^{10})$
   (e) $T_6 = -2$, $T_{20} = -\frac{1}{2^{13}}$, $S_{10} = \frac{128}{3}(1 + 2^{-10})$

3. (a) $a = 506$, $d = -18$ (b) $81(1 - (\frac{1}{3})^5)$ (c) $T_2 = (\frac{9}{4})^3$ (d) $n^2 + 6n$

4. (a) $437,988.84 (b) 5 metres