Worksheet 3.5 Simultaneous Equations

Section 1 Number of Solutions to Simultaneous Equations

In maths we are sometimes confronted with two equations in two variables and we want to find out which values of the variables satisfy both of the equations. Sometimes there will be no values of the variables that allow both equations to hold, and other equations will have many possible values of the variables. The process of finding solutions is called solving simultaneous equations .

For example, we might be asked to find the x and y values that allow both of the following equations to be true:

$$\begin{aligned} 5x + 2y &= 3 \\ 4x + 2y &= 4 \end{aligned}$$

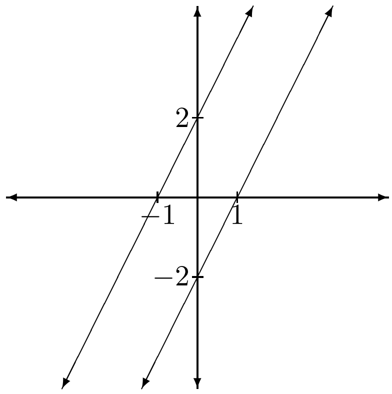
Recall from worksheet 2.10 that the general equation of a line can be written in the form $ax + by + c = 0$, for a, b, and c constants. Both of the equations above have the correct form for the equation of a line. This is always the case when solving linear simultaneous equations in two variables. This means that solving simultaneous equations is the same as finding the point of intersection of lines. If certain values of x and y satisfy both equations, the point (x, y) will lie on both the lines. If we think about a system of simultaneous equations as representing lines on the cartesian plane we can tell how many solutions there will be to the equations without actually solving them. When we draw two lines on the plane, there are three possibilities:

1. The lines cross just once.
2. The lines never cross.
3. The lines lie on top of each other.

The first case corresponds to a unique solution, i.e. there is only one value for each variable that will satisfy both equations. The second case occurs when the two lines are parallel, and aren't touching. Parallel lines have the same slope (gradient). For instance, the two lines

$$\begin{aligned} 1.\quad y &= 2x + 2 \quad \text{and} \\ 2.\quad y &= 2x - 2 \end{aligned}$$

are parallel, and don't touch. They have the same gradient, in this case 2, but the y-intercepts are different. Consequently, they two lines never touch each other. Equation 1 has a y-intercept of 2, and equation 2 has a y-intercept of -2. If we draw these lines, we get



The third case arises when the two equations represent the same line on the plane, and so touch everywhere. The two equations

1. $y = 3x + 6$ and
2. $2y = 6x + 12$

lie on top of one another when graphed because if we take equation 2 and divide both sides by 2, they we get equation 1 exactly. If two lines lie on top of one another there are an infinite number of (x, y) pairs that will satisfy both equations. Namely, every pair (x, y) that satisfies equation 1 will also satisfy equation 2.

To check on the number of solutions to a system of simultaneous equations we can rearrange both equations to the slope-intercept form and then compare gradients and intercepts. If the gradients are different, we will have a single (i.e. unique) solution. If the gradients are the same, but the y-intercepts are different, then we will have no solutions. If the gradients are the same, and the y-intercepts are the same, then there will be an infinite number of solutions.

Putting this algebraically, if we have:

$$\begin{aligned} y &= m_1 x + b_1 \\ y &= m_2 x + b_2 \end{aligned}$$

Then

- If $m_1 \neq m_2$ then there is one solution.
- If $m_1 = m_2$ and $b_1 \neq b_2$ there are no solutions.
- If $m_1 = m_2$ and $b_1 = b_2$ there is an infinite number of solutions

Example 1 : How many solutions do the following simultaneous equations have?

1. $3y + 6x = 9$
2. $2y + 10x = 4$

Rearranging, we get

$$\begin{aligned} y &= -2x + 3 \quad \text{for equation 1} \\ y &= -5x + 2 \quad \text{for equation 2} \end{aligned}$$

The gradients of the two lines are different, so there will be one solution.

Example 2 : How many solutions do the following simultaneous equations have?

1. $5y + 10x = 5$
2. $y + 2x = 2$

Rearranging, we get

$$\begin{aligned} y &= -2x + 1 \quad \text{for equation 1} \\ y &= -2x + 2 \quad \text{for equation 2} \end{aligned}$$

The gradients of the two lines are the same, but the intercepts are different. Then the lines are parallel, but don't touch. There are no solutions to the system.

Example 3 : How many solutions do the following simultaneous equations have?

1. $5 + 10x = 2y$
2. $4y - 20x = 10$

Rearranging, we get

$$\begin{aligned} y &= \frac{5}{2} + 5x \quad \text{for equation 1} \\ y &= \frac{5}{2} + 5x \quad \text{for equation 2} \end{aligned}$$

The gradients of the two lines are the same, and the intercepts are also the same. Then the lines are on top of each other, and there are infinitely many solutions.

Exercises:

1. How many solutions would each of the following pairs of equations have?

   (a) $y = 2x + 1$
   $y = 3x - 2$

   (b) $2y = 6x - 4$
   $y = 3x - 2$

   (c) $y + x - 2 = 0$
   $y - x + 1 = 0$

2. Check your answers by graphing the pairs of lines on a number plane.

Section 2 Solving simultaneous equations

The previous section discussed how many solutions there are to a system of 2 simultaneous equations in 2 unknowns (which we have been writing as x and y). We will learn how to find solutions to a system of simultaneous equations by example. Given two equations and two unknowns, our objective is to reduce this to one equation and one unknown, which we know we can solve.

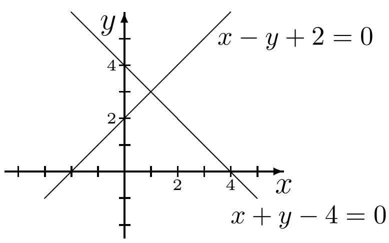
We can solve two equations simultaneously by graphing them and finding their point of intersection.

Let us solve for the following system graphically:

(i) $x + y - 4 = 0$

(ii) $x - y + 2 = 0$

Drawing the two lines on one graph, we get:



From the graph, we can see that the point of intersection is $(1, 3)$. Substituting $x = 1$ and $y = 3$ into the two equations, we see:

$$\begin{array}{rrcll} \text{(i)} & 1+3-4 & = & 0 & \text{(true)} \\ \text{(ii)} & 1-3+2 & = & 0 & \text{(true)} \end{array}$$

Hence the point $(1, 3)$ satisfies both equations. Sometimes the point of intersection is not easy to read off the graph, so solving a system of equations algebraically is often easier and more precise.

Example 1 : Solve the system:

(i) $x + y = 4$

(ii) $x - y = -2$

If we add equation (i) to equation (ii) the y's will cancel:

$$\begin{array}{lrcl} \text{(i) + (ii)} & 2x & = & 2 \\ & x & = & 1 \end{array}$$

We now substitute $x = 1$ into either equation (i) or (ii). Let us choose equation (i). Then

$$\begin{array}{rcl} 1+y & = & 3 \\ y & = & 3 \end{array}$$

So the solution is $x = 1$ and $y = 3$. (You can, and should, check the solution by substituting the values of x and y into both equations (i) and (ii).)

Example 2 : Solve the system

$$\begin{array}{rrcl} \text{(i)} & 3x+2y & = & 5 \\ \text{(ii)} & x+2y & = & -3 \end{array}$$

There are the same number of y's i equation (i) and (ii), so if we subtract the equations we would eliminate the y's.

$$\begin{array}{lrcll} \text{(i) - (ii)} & 2x & = & 8 & (\text{Note } 5-(-3)=8) \\ & x & = & 4 & \end{array}$$

Substitute $x = 4$ in equation (ii):

$$\begin{aligned} 4 + 2y &= -3 \\ 2y &= -7 \\ y &= -3\frac{1}{2} \end{aligned}$$

The solution is $x = 4$, $y = -3\frac{1}{2}$; check that the solution satisfies (i) and (ii).

<u>Example 3</u> : Solve the system

$$\begin{array}{lrcl} (1) & y &=& 2x + 3 \\ (2) & y &=& 2x + 5 \end{array}$$

Subtracting (2) from (1) gives

$$(3) \qquad 0 = -2$$

This is nonsense, and a check shows that there is no solution to this system because the lines that the equations represent are parallel.

<u>Example 4</u> : Solve the system

$$\begin{array}{lrcl} (1) & 2x + 3y &=& 10 \\ (2) & 5x + 4y &=& 11 \end{array}$$

Sometimes it is necessary to multiply both equations by different numbers to get the same multiple of one of the variables. Here is another example of the usefulness of being able to find the lowest common multiple of two numbers. We will take equation (1) times 5, and equation (2) times 2:

$$\begin{array}{lrcl} (3) & 10x + 15y &=& 50 \\ (4) & 10x + 8y &=& 22 \end{array}$$

Now subtracting (4) from (3):

$$\begin{aligned} 10x + 15y - (10x + 8y) &= 50 - 22 \\ 7y &= 28 \end{aligned}$$

which gives $y = 4$. Substituting this back into equation (1) gives $x = -1$. A check reveals that $x = -1$ and $y = 4$ is indeed a solution to the original equations.

Exercises:

1. Solve graphically the system of equations

$$\begin{aligned} x + y &= 3 \\ x - y &= 1 \end{aligned}$$

2. Solve the following systems algebraically where possible.

   (a) $$\begin{aligned} 2x - y &= 5 \\ 3x + y &= 10 \end{aligned}$$

   (b) $$\begin{aligned} 2x + 3y &= -1 \\ 2x + y &= 4 \end{aligned}$$

   (c) $$\begin{aligned} 3x + 2y &= 4 \\ x + y &= 5 \end{aligned}$$

   (d) $$\begin{aligned} 2x + 5y &= -4 \\ 3x + 2y &= -6 \end{aligned}$$

Exercises 3.5 Simultaneous Equations

1. How many solutions (one, none, or infinite) will each of the following pairs of equations have?

   (a) $$\begin{aligned} 2x + 3y &= 5 \\ 2x + 3y &= 9 \end{aligned}$$

   (b) $$\begin{aligned} x + 2y &= 3 \\ x - 2y &= 3 \end{aligned}$$

   (c) $$\begin{aligned} x - 2y &= -1 \\ -2x + 4y &= 2 \end{aligned}$$

   (d) $$\begin{aligned} x + 5 &= 3y \\ 4x + y &= 2 \end{aligned}$$

   (e) $$\begin{aligned} y &= \tfrac{1}{3}x + 9 \\ x - 3y &= 2 \end{aligned}$$

2. Solve the following systems of equations.

   (a) $$\begin{aligned} y &= 2x - 3 \\ y &= x + 5 \end{aligned}$$

   (b) $$\begin{aligned} 2x - y &= 4 \\ x + 2y &= 3 \end{aligned}$$

   (c) $$\begin{aligned} 2x - 3y &= 1 \\ 3x + 4y &= -1 \end{aligned}$$

3. The sum of Peter and Anneka's ages is 24, and the difference between their ages is 6. Find their ages given that Peter is older than Anneka.

Answers 3.5

1. (a) None
   (b) One
   (c) Infinite
   (d) One
   (e) None

2. (a) $x = 8$, $y = 13$
   (b) $x = \frac{11}{5}, y = \frac{2}{5}$
   (c) $x = \frac{1}{17}$, $y = -\frac{5}{17}$

3. Peter is 15, Anneka is 9