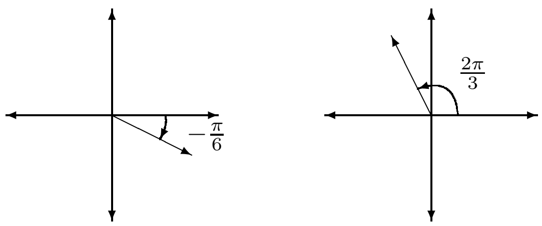
Worksheet 3.4 Further Trigonometry

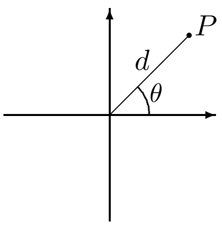
Section 1 Trig ratios for angles of any magnitude

Recall from the last worksheet how we described a way of drawing angles of any magnitude on the cartesian plane. If we use the positive x-axis to represent our starting point, then rotate this axis in an anticlockwise direction through α radians, we have an angle of α radians (with α positive). A negative angle can be drawn by rotation in a clockwise direction.

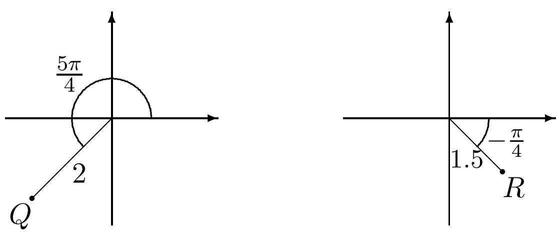
<u>Example 1</u> : We draw the angles $-\frac{\pi}{6}$ and $\frac{2\pi}{3}$.



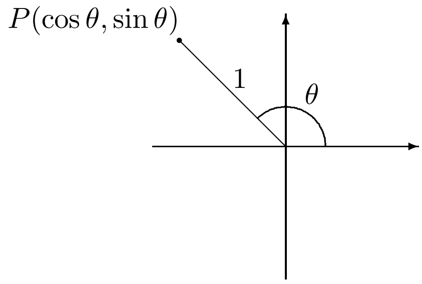
Instead of representing a point P using x and y coordinates, we could represent it as an angle of rotation and a distance away from the origin.



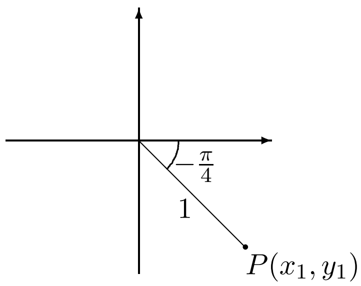
As examples consider two points: Q, which is 2 units away from the origin and rotated through an angle of $\frac{5\pi}{4}$; R, which is 1.5 units away from the origin and rotated through an angle of $-\frac{\pi}{4}$.



The question now is how to connect this method of specifying a point with the usual way of using the x and y coordinates. In a previous section we defined $\sin\theta$ and $\cos\theta$ using right angled triangles. A much more useful definition is the following. Let a point P be exactly one unit away from the origin, and rotated by an angle θ. Then the x coordinate of P is defined to be $\cos\theta$, and the y coordinate is defined to be $\sin\theta$. The illustration of the definition is:



Example 2 : Draw a picture to determine whether $\sin(-\frac{\pi}{4})$ and $\cos(-\frac{\pi}{4})$ are positive or negative.

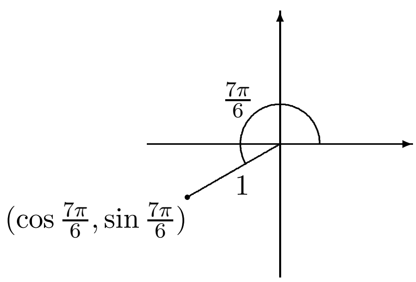


From the definitions of sine and cosine we have

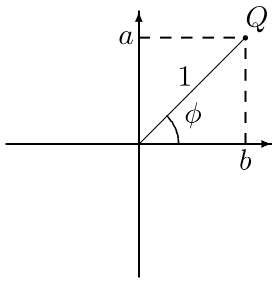
$$x_1 = \cos(-\frac{\pi}{4})$$
$$y_1 = \sin(-\frac{\pi}{4})$$

It is apparent from the picture that $\cos(-\frac{\pi}{4}) > 0$ and $\sin(-\frac{\pi}{4}) < 0$.

Example 3 : Draw a picture to locate the point $(\sin(\frac{7\pi}{6}), \cos(\frac{7\pi}{6}))$.



Notice what happens when we apply this definition to an angle that is between 0 and $\frac{\pi}{2}$. Let Q be the point that is exactly one unit away from the origin, and rotated by an angle ϕ, where $0 < \phi < \frac{\pi}{2}$. Say the x coordinate of Q is b and the y coordinate is a. The relevant picture is:

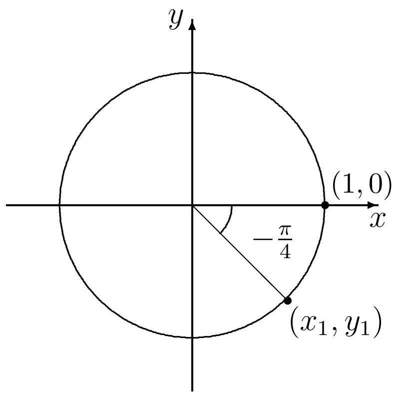


If you think about the right angled triangle formed be the points $(0, 0)$, $(b, 0)$, and (a, b) and apply the right angled definitions of sine and cosine then you get the formulae

$$\sin\phi = \frac{a}{1} = a$$
$$\cos\phi = \frac{b}{1} = b$$

which is to say the x and y coordinates of the point Q are $\cos\phi$ and $\sin\phi$ respectively. This is exactly the definition that we have just proposed! The point is that the definitions of sine and cosine that we have seen before, in terms of right angled triangles, match the new definition that we have just given in the case that the angles are between 0 and $\frac{\pi}{2}$ ($0°$ and $90°$).

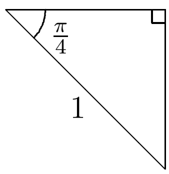
The advantage with the new definition is that it allows us to find the sine and cosine of angles of any magnitude, as well as for negative angles. We do this by drawing the unit circle (which is a circle of radius 1 centred on the origin). Any point on the circle is then exactly one unit away from the origin. Now, drawing in our angle from example 2, $-\frac{\pi}{4}$, we get



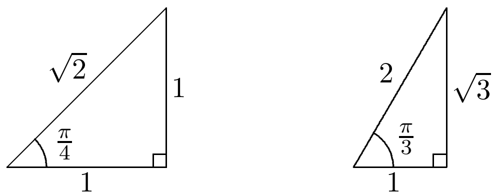
Now,

$$x_1 = \cos(-\frac{\pi}{4})$$
$$y_1 = \sin(-\frac{\pi}{4})$$

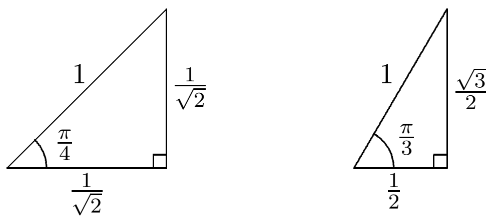
Notice that the triangle defined by the points $(0, 0)$, $(x_1, 0)$ and (x_1, y_1) is a right angled triangle; the hypotenuse is of length 1 because the the radius of the unit circle is of length 1. It is drawn here:



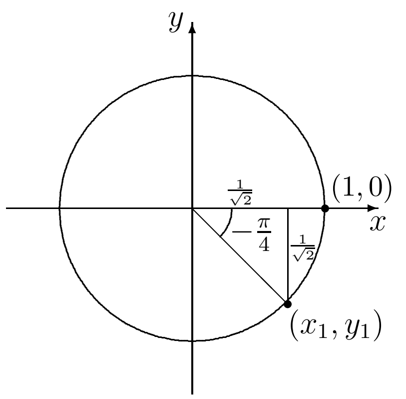
What remains is to find the length of the two sides of the triangle, which we will do by recalling our standard triangles from an earlier worksheet and using the properties of similar triangles. The two standard triangles we looked at were



We can make the triangle on the left the same as the one that we took out of the unit circle by dividing all the lengths by a factor of $\sqrt{2}$. Similarly, we can make the standard triangle on the right have a hypotenuse of length 1 by dividing each side by 2.



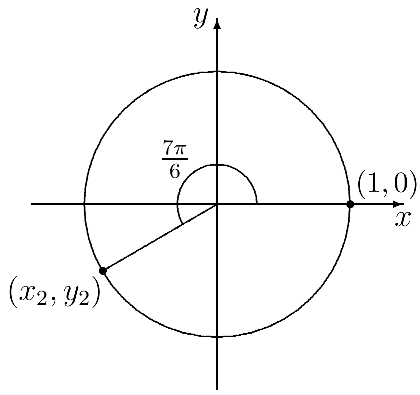
Putting the lengths back onto the unit circle picture gives us



Then we have

$$x_1 = \cos(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$
$$y_1 = \sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$$

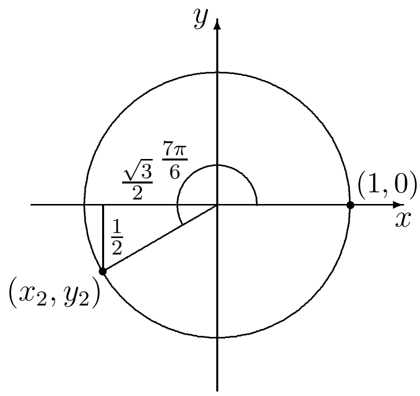
Example 4 : Calculate the sine and cosine of $\frac{7\pi}{6}$.



Recalling the definitions of sine and cosine, we have

$$
\begin{aligned}
x_2 &= \cos(\frac{7\pi}{6}) \\
y_2 &= \sin(\frac{7\pi}{6})
\end{aligned}
$$

By extracting the right angled triangle which connects the points $(0, 0)$, $(x_2, 0)$, and (x_2, y_2) and comparing it to the scaled standard triangles, we can put the following distances onto the unit circle diagram.



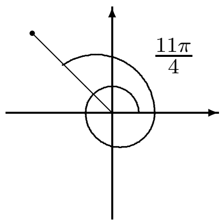
As a result, we get

$$
\begin{aligned}
x_2 &= \cos(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2} \\
y_2 &= \sin(\frac{7\pi}{6}) = -\frac{1}{2}
\end{aligned}
$$

What happens if an angle is bigger than 2π or less than -2π? Since a full revolution of a circle is 2π radians, the position on the circle is unchanged if we go an angle of θ or an angle of

$\theta + 2\pi$. If the position on the circle is unchanged by adding an angle of 2π, and the sines and cosines are defined in terms of the coordinates of appropriate points on the circle, then the sines and cosines of angles are unchanged by adding or subtracting multiples of 2π radians.

As an example, graph the angle $\frac{11\pi}{4}$ on the cartesian plane.



Notice that we would end up with an angle pointing in the same direction if we had performed a rotation of $\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}$.

Exercises:

1. Find the exact ratio for each of the following

   (a) $\sin \frac{\pi}{4}$ (b) $\tan(-\frac{\pi}{4})$ (c) $\cos \frac{\pi}{6}$ (d) $\cos \frac{7\pi}{6}$ (e) $\sin(-\frac{\pi}{3})$ (f) $\tan \frac{3\pi}{4}$ (g) $\cos \frac{3\pi}{4}$ (h) $\sin(-\frac{2\pi}{3})$ (i) $\tan \frac{5\pi}{6}$ (j) $\cos(-\frac{\pi}{3})$

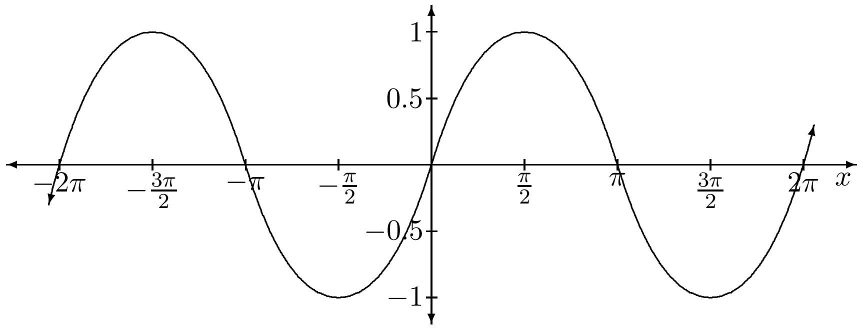
2. Use a calculator to find the following to 2 decimal places.

   (a) $\cos 1.6$ (b) $\sin \frac{\pi}{8}$ (c) $\tan \frac{7\pi}{6}$ (d) $\cos(-\frac{\pi}{7})$ (e) $\sin(-0.6)$ (f) $\sin \frac{\pi}{9}$ (g) $\cos \frac{8\pi}{5}$ (h) $\tan(-\frac{\pi}{9})$

Section 2 Graphs of Trig Functions

The trig functions can be graphed on a Cartesian plane as functions of x. The unit of measurement for x is radians. It is helpful to be able to recognize the graphs of the main trig functions.

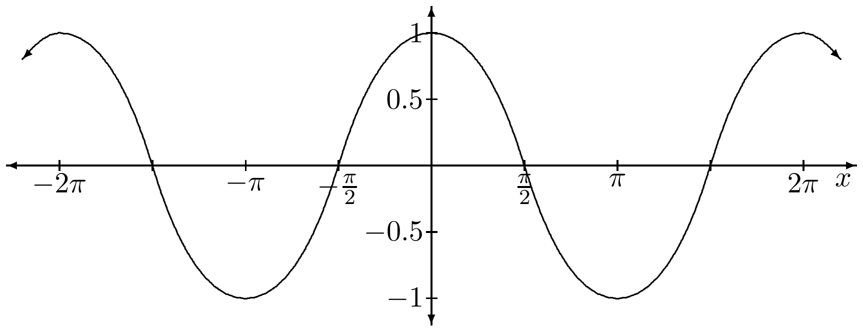
The function $y = \sin x$ is odd, with an x-intercept every integer multiple of π. It looks like this:



The function $y = \sin x$ is also periodic, with period 2π. This means that $\sin(x) = \sin(x + 2\pi)$ for all values of x. If a function $f(x)$ is periodic with period b, then $f(x) = f(x + b)$ for all x.

We can see from the graph of $y = \sin x$ that the range of the function is $[-1, 1]$ i.e. $-1 \leq \sin x \leq 1$ for all x.

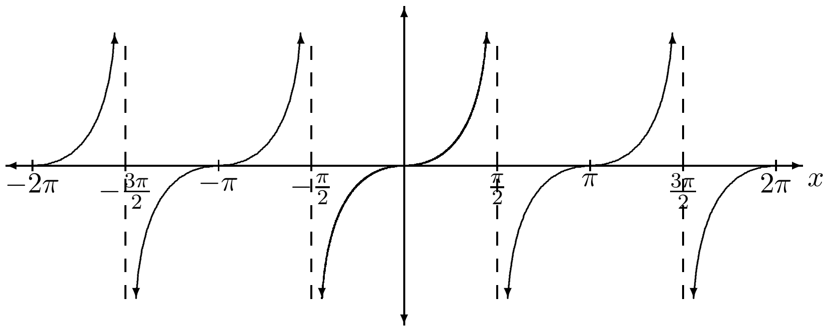
The function $y = \cos x$ is even. It is also periodic with period 2π. The y-intercept is 1 and the x-intercepts are at $\frac{\pi}{2} + k\pi$ for integer k. The graph of $y = \cos x$ looks like this:



Notice that the range is also $[-1, 1]$, so $-1 \leq \cos x \leq 1$ for all x. The graphs of $\sin x$ and $\cos x$ will help you to remember the values of $\sin x$ and $\cos x$ for $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π if you are a visual person. Some people find it easier to remember the pictures, other people the numbers. From the graph we get

$$
\begin{aligned}
\sin(-\pi) &= 0 & \qquad \cos(-\pi) &= -1 \\
\sin(-\tfrac{\pi}{2}) &= -1 & \cos(-\tfrac{\pi}{2}) &= 0 \\
\sin(\tfrac{\pi}{2}) &= 1 & \cos(\tfrac{\pi}{2}) &= 0 \\
\sin(\tfrac{3\pi}{2}) &= -1 & \cos(\tfrac{3\pi}{2}) &= 0 \\
\sin(2\pi) &= 0 & \cos(2\pi) &= 1
\end{aligned}
$$

The graph of $y = \tan x$ looks completely different from either $\cos x$ or $\sin x$. It is a periodic function with period π and it looks like this:



Notice that the x-intercepts are integer multiples of π, and that the y-intercept is 0. Notice also that $y = \tan x$ is not defined at $\frac{\pi}{2} + k\pi$ for any integer k. Recall that

$$\tan x = \frac{\sin x}{\cos x}$$

so $\tan x$ is undefined when $\cos x = 0$, which is at $\frac{\pi}{2} + k\pi$ for any integer k.

Exercises:

1. (a) Given $y = 2\sin x$, complete the table of values

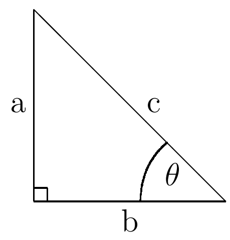
x	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π
y									

   (b) Using the table draw the graph of $y = 2\sin x$ for $-\pi \leq x \leq \pi$.

Section 3 Pythagorean Identities

There are some equalities known as trigonometric identities which are very useful in solving some kinds of problems. The first one that we look at is derived from Pythagoras' theorem. Recall:

$$\begin{aligned} \sin\theta &= a/c \\ \cos\theta &= b/c \\ a^2+b^2 &= c^2 \end{aligned}$$



From the above relations, we then have:

$$\begin{aligned} \sin^2\theta+\cos^2\theta &= (\frac{a}{c})^2+(\frac{b}{c})^2 \\ &= \frac{a^2}{c^2}+\frac{b^2}{c^2} \\ &= \frac{a^2+b^2}{c^2} \\ &= \frac{c^2}{c^2} \\ &= 1 \end{aligned}$$

Then we have that for any angle θ:

$$\boxed{\sin^2\theta+\cos^2\theta=1}$$

The next two identities are also important, but will not be derived. For any angles A and B:

$$\boxed{\begin{aligned} \sin(A+B) &= \sin A\cos B+\sin B\cos A \\ \cos(A+B) &= \cos A\cos B-\sin A\sin B \end{aligned}}$$

These identities can be used to find the cos and sin of any angles bigger than $\frac{\pi}{2}$. We can derive more trig identities from the ones that we already have.

Example 1 :

$$\begin{aligned} \sin 2x &= \sin(x+x) \\ &= \sin x\cos x+\sin x\cos x \\ &= 2\sin x\cos x \end{aligned}$$

Example 2 :

$$\begin{aligned}\cos 2x &= \cos(x+x)\\ &= \cos x\cos x - \sin x \sin x\\ &= \cos^2 x - \sin^2 x\end{aligned}$$

Recall that $y = \cos x$ is an even function, therefore

$$\cos(-x) = \cos(x)$$

Recall that $y = \sin x$ is an odd function, therefore

$$\sin(-x) = -\sin(x)$$

Example 3 :

$$\begin{aligned}\sin(A-B) &= \sin A\cos(-B) + \sin(-B)\cos A\\ &= \sin A\cos B - \sin B\cos A\end{aligned}$$

Example 4 :

$$\begin{aligned}\cos(A-B) &= \cos A\cos(-B) - \sin(-B)\sin A\\ &= \cos A\cos B + \sin B\sin A\end{aligned}$$

These identities can be used in many ways. One use for them is an alternative way of finding the trig ratios of angles between 0 and 2π.

Example 5 : Calculate the sine and cosine of $\frac{7\pi}{6}$.

$$\begin{aligned}\sin\frac{7\pi}{6} &= \sin(\pi+\frac{\pi}{6})\\ &= \sin\pi\cos\frac{\pi}{6} + \cos\pi\sin\frac{\pi}{6}\\ &= 0\times\frac{\sqrt{3}}{2} + (-1)\times\frac{1}{2}\\ &= -\frac{1}{2}\\ \cos\frac{7\pi}{6} &= \cos(\pi+\frac{\pi}{6})\\ &= \cos\pi\cos\frac{\pi}{6} - \sin\pi\sin\frac{\pi}{6}\\ &= (-1)\times\frac{\sqrt{3}}{2} + (0)\times\frac{1}{2}\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Writing the angle $\frac{7\pi}{6}$ as $\pi + \frac{\pi}{6}$ wasn't the only option – we could have used $\frac{7\pi}{6} = \frac{3\pi}{2} - \frac{\pi}{3}$. (Notice that the answers that we have here agree with the values calculated using the unit circle earlier in the worksheet.)

Exercises:

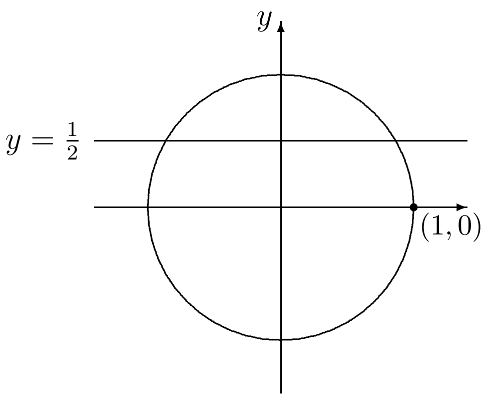
1. Use exact ratios to show that
$$\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{6} = 1$$

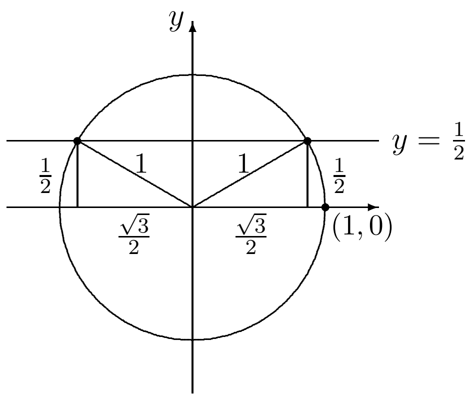
2. Use exact values to show that equation
$$\sin(A+B) = \sin A \cos B + \sin B \cos A$$
is satisfied when $A = 0$ and $B = \frac{\pi}{3}$.

Section 4 Solving trigonometric equations

In the previous section θ was given and we evaluated the trigonometric ratios for the angle. Now we investigate the situataion where we must find the value, or values, of θ when we are given a trigonometric ratio.

Example 1 : Solve $\sin\theta = \frac{1}{2}$ for $0 \leq \theta \leq 2\pi$. Recall that $\sin\theta$ is the y coordinate of a point on the unit circle. The first step then will be to draw a unit circle and draw the line $y = \frac{1}{2}$; the next, and last, step is to determine the angles of the points on the unit circle where the line $y = \frac{1}{2}$ cuts. We draw two pictures, one with the basic information we have just outlined, and one with a few distances that have been worked out.

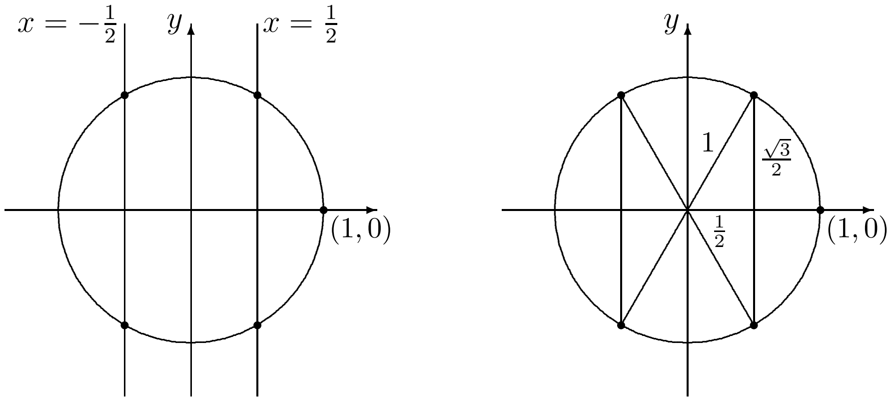




The first thing to note is that there are two solutions. The lengths shown have been figured out using the fact that the vertical distances are $\frac{1}{2}$, the fact that the radius of the circle is 1, and by recognizing that the triangles hidden in the picture are scaled versions of the standard triangles (which are shown in section 1). Given that we know the angles in the standard triangles, we can read off the angles to the two solutions as $\frac{\pi}{6}$ and $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Example 2 : Solve $\cos^2\theta = \frac{1}{4}$.

This must have solutions given by $\cos\theta = \frac{1}{2}$ and $\cos\theta = -\frac{1}{2}$. The definition of $\cos\theta$ is that it is the x coordinate of the point defined by some angle around the unit circle. So the solution will be obtained by drawing the lines $x = \frac{1}{2}$ and $x = -\frac{1}{2}$, locating the points of intersection with the unit circle, then finding the appropriate angles. Again we draw two pictures: one with minimal information, so we can see roughly where the solutions are as well as how many solutions there are; the other picture has details of distances and so on.



From the first picture we can see that there are four solutions, as well as the fact that there is one solution in each quadrant. Of the four triangles in the second picture, only the distances for one of them have been shown, as all the other triangles are similar. The distances have again been found by scaling a standard triangle from section 1. As the angles of the standard triangle are known, so are the angles of the four points shown. They are $\frac{\pi}{3}$, $\pi - \frac{\pi}{3}$, $\pi + \frac{\pi}{3}$, and $2\pi - \frac{\pi}{3}$. (Another way to write the solutions would be $\frac{\pi}{3}$, $\pi - \frac{\pi}{3}$, $-\frac{\pi}{3}$, and $-(\pi - \frac{\pi}{3})$.)

Exercises:

1. Solve the following equations for θ; restrict your answers to $0 \leq \theta \leq 2\pi$.

   (a) $\sin\theta = -\frac{\sqrt{3}}{2}$

   (b) $\tan\theta = -\sqrt{3}$

   (c) $\cos\theta = \frac{1}{\sqrt{2}}$

   (d) $\cos\theta = \frac{\sqrt{3}}{2}$

   (e) $\tan\theta = -1$

   (f) $\sin\theta = -\frac{1}{2}$

Exercises 3.4 Further Trigonometry

1. Find the exact ratios of

   (a) $\sin \frac{3\pi}{4}$

   (b) $\tan \frac{\pi}{6}$

   (c) $\cos \frac{7\pi}{4}$

   (d) $\cos \frac{4\pi}{3}$

   (e) $\sin \frac{5\pi}{6}$

   (f) $\tan -\frac{\pi}{3}$

2. (a) Use the expansion $\sin(A + B) = \sin A \cos B + \sin B \cos A$ to find the exact value of $\sin \frac{7\pi}{12}$. Note that $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$.

   (b) Use the expansion $\cos(A - B) = \cos A \cos B + \sin A \sin B$ to find the exact value of $\cos \frac{\pi}{12}$. Note that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.

3. Solve for θ in the interval $0 \leq \theta \leq 2\pi$.

   (a) $\tan \theta = \sqrt{3}$

   (b) $\sin \theta = \frac{1}{\sqrt{2}}$

   (c) $\cos \theta = -\frac{1}{\sqrt{2}}$

Answers 3.4

1. (a) $\frac{1}{\sqrt{2}}$
   (b) $\frac{1}{\sqrt{3}}$
   (c) $\frac{1}{\sqrt{2}}$
   (d) $-\frac{1}{2}$
   (e) $\frac{1}{2}$
   (f) $-\sqrt{3}$

2. (a) $\frac{\sqrt{3}+1}{2\sqrt{2}}$
   (b) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

3. (a) $\frac{\pi}{3}$, $\frac{4\pi}{3}$
   (b) $\frac{\pi}{4}$, $\frac{3\pi}{4}$
   (c) $\frac{3\pi}{4}$, $\frac{5\pi}{4}$