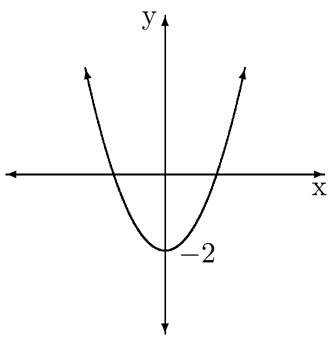
Worksheet 3.2 Graphs

Section 1 Range & Domain

In the last worksheet we mentioned that functions can be represented as graphs. Graphs have already been referred to in worksheet 2.10 when we looked at graphing a straight line. The graph of a function is the collection of all points (x, y) that satisfy a given function. From looking at graphs we can learn a lot of information about the function it represents. From a graph, we can make estimates about the value of the function at certain inputs; we can see where maximum and minimum values of the function are; we can see how rapidly the function is increasing or decreasing.

Two important pieces of information we can read off a graph are the range and domain of the function. The range of a function is all the values that the function takes. So if y is a function of x, the range is all the y-values that can be taken. The range may be written in one of several ways - typically as an interval, or using inequality signs.

Example 1 :



The arrows on the graph of $f(x)$ indicate that it keeps going upwards. The range of $f(x)$ can be written as

$$-2 \leq y$$

That is, y can be greater than or equal to -2.

In interval notation this is written as $[-2, \infty)$.

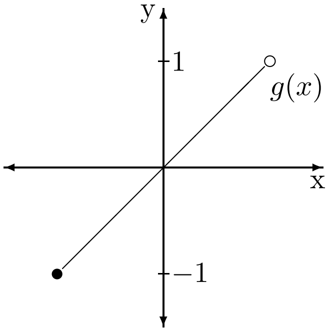
Note: The square bracket indicates that -2 is included in the range. The round bracket indicates all numbers up to but not including the end part. So

$$(2, \infty) \quad \text{is the same as} \quad y > 2$$

and

$$[1, 3) \quad \text{is the same as} \quad 1 \leq y < 3$$

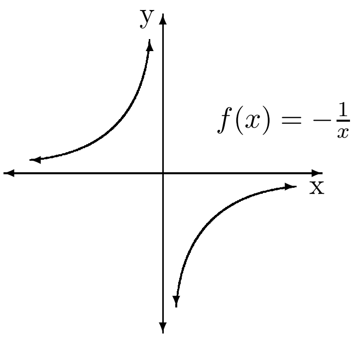
Example 2 :



The open circle on a graph means the same as an open circle on a number line: all numbers up to but not including the point. Thus the range of $g(x)$ is $-1 \leq y < 1$, or in interval notation $[-1, 1)$.

The domain of a function is all the inputs that make sense. In other words, for a function $f(x)$ it is all the x-values for which the function is defined. The domain of a function is normally written in the same notation as the range i.e. either using inequalities or interval notation.

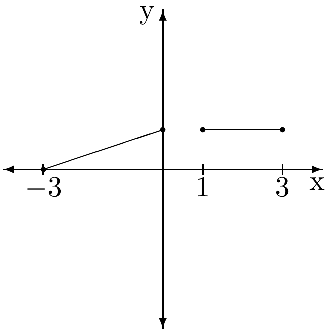
Example 3 :



The domain is $x > 0$ and $x < 0$. Or equivalently, $(-\infty, 0)$ and $(0, \infty)$. $f(x)$ does not make sense for $x = 0$, so it is not included in the domain.

The range of this function is $y > 0$ and $y < 0$. Or, equivalently, $(-\infty, 0) \cup (0, \infty)$.

Example 4 :

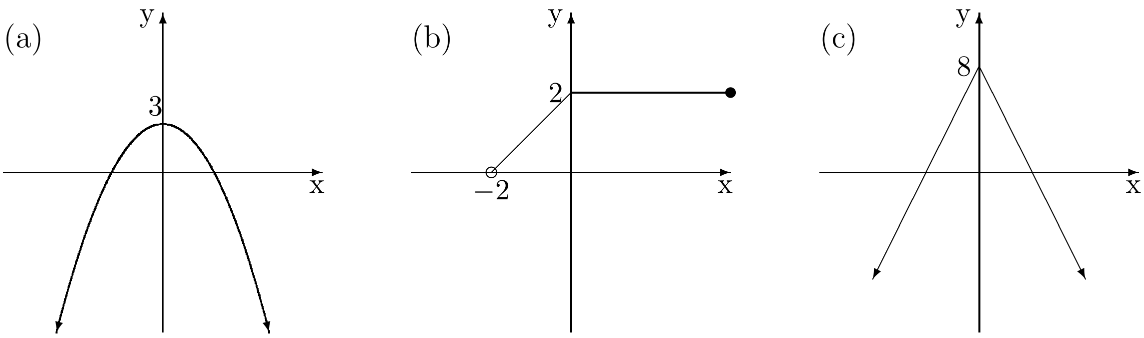


The domain of this graph is $-3 \leq x \leq 0$ and $1 \leq x \leq 3$, or in interval notation $[-3, 0]$ and $[1, 3]$.

Exercises:

1. Find the range and domain of the following graphs

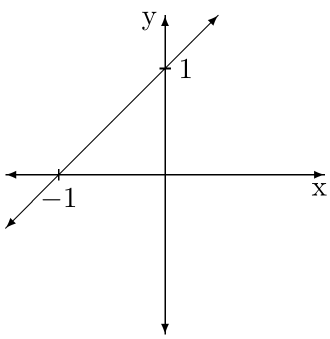
(a) (b) (c)



Section 2 Intercepts and Reading Graphs

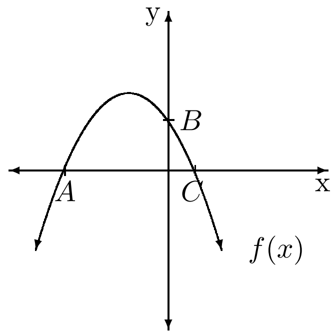
The value of a function when the input is zero (i.e. the y value when $x = 0$) is called the y-intercept. This is where the function crosses the y-axis. The input which gives the function a value of zero (i.e. the x values that give $y = 0$) are called the x-intercepts. They are called the x-intercepts because this is where the plot of the function crosses the x-axis. To find the y-intercept, we simply substitute $x = 0$ into the function. The output is the y-intercept. To find the x-intercept, we let $y = 0$ and then solve the equation for x. This may not always be a simple procedure. The intercepts give us the beginning of a picture of the function and will help us to represent the function graphically.

Example 1 :



The y-intercept is $+1$. The x-intercept is -1. i.e. when $x = 0$, $y = 1$ and when $y = 0$, $x = -1$.

Example 2 :

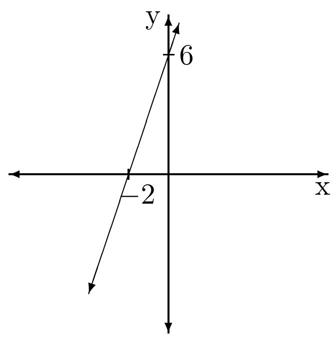


$f(x)$ has a zero value at A and C; B marks the y-intercept. There are many possible formulae for the graph, and one possibility is a parabola.

Example 3 : Let $y = 3x + 6$. Then at $x = 0$, $y = 3 \times 0 + 6 = 6$. So the y-intercept is 6. Now set $y = 0$ to find the x-intercept.

$$\begin{aligned} 0 &= 3x + 6 \\ x &= -2 \end{aligned}$$

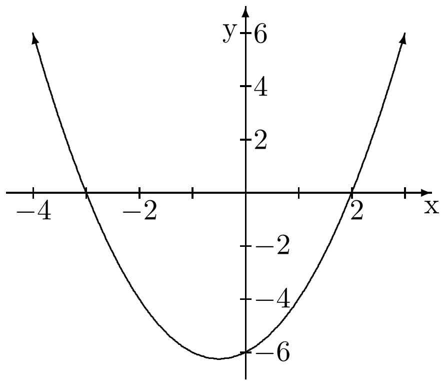
So the x-intercept is -2. Since $y = 3x+6$ is the equation for a straight line, we can now draw the graph. The intercepts of a straight line give us enough information to draw the graph (unless both intercepts are at the origin)



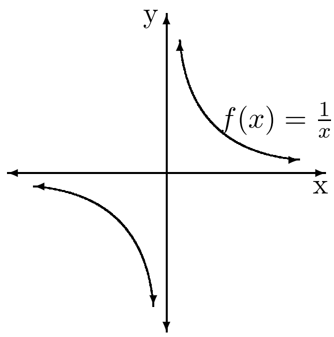
Example 4 : Let $y = x^2 + x + -6$. The y-intercept is -6 since $y = -6$ when $x = 0$. To find the x-intercepts, one must solve the equation

$$\begin{aligned} 0 &= x^2 + x - 6 \\ 0 &= (x + 3)(x - 2) \end{aligned}$$

This implies that we can have either $(x+3) = 0$ which gives $x = -3$ or $(x-2) = 0$ which gives $x = 2$. The x-intercepts are then -3 and 2. The equation we are currently dealing with is called a quadratic, and the intercepts don't give enough information to plot the graph. Since all quadratic functions are symmetrical, the turning point will always occur half-way between the x-intercepts. The equation $y = x^2 + x - 6$ is that of a parabola.

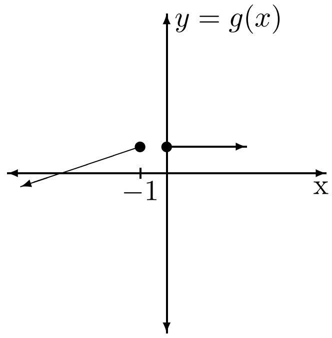


The worksheet on differentiation gives us another method for finding the coordinates of A. Point A is called a turning point because the graph changes direction at point A. Another important output of functions that can be seen from graphs are the values of y and x where the function isn't defined. These will appear as breaks in the graph. For example, we noted in the last worksheet that the function $y = \frac{1}{x}$ is not defined at $x = 0$. Also, there is no input that will give an output of $y = 0$, i.e. there are no x or y-intercepts. The graph of $y = \frac{1}{x}$ looks like this:



There is a break in the graph of this hyperbola at $x = 0$ and at $y = 0$.

Example 5 :



The function $g(x)$ is not defined for $-1 < x < 0$.

Exercises:

1. Graph the following equations of straight lines by first finding x and y intercepts.
   (a) $y = x + 2$
   (b) $y = x - 3$
   (c) $y = 2x + 4$
   (d) $x + y - 6 = 0$
   (e) $y = 4 - x$

Section 3 Odd and Even Functions

Some functions can be classed as odd or even functions. Many functions, however, are not odd or even. If we know that a certain function is odd or even, it will help us draw the graph.

An even function is one in which

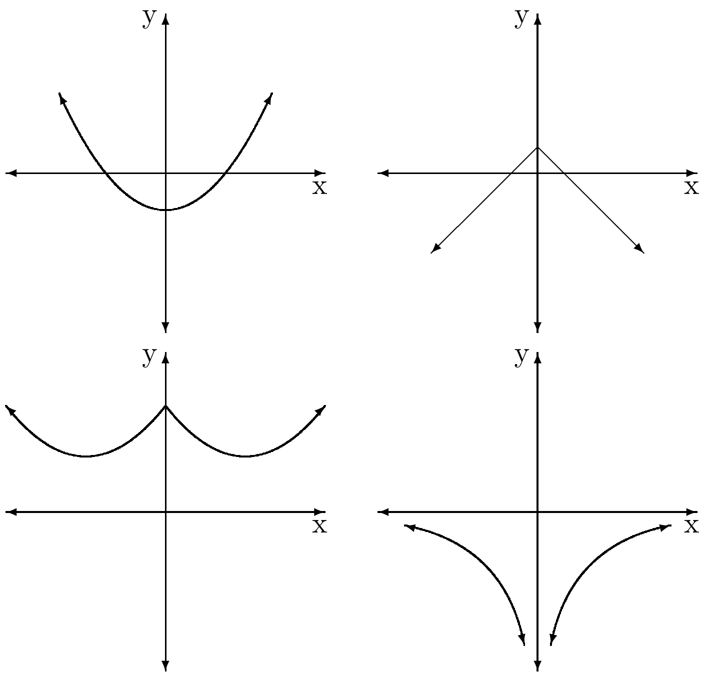
$$\boxed{f(a) = f(-a)}$$

for all a. That is,, whatever number we choose as input, the output of the function will be the same if we change the sign of the input. For an even function, if an input of $-c$ gives an output of b, then the input c also gives an output of b. In function notation, this says that if $f(-c) = b$, then $f(c) = b$ also.

Example 1 : The function $f(x) = x^2$ is even.

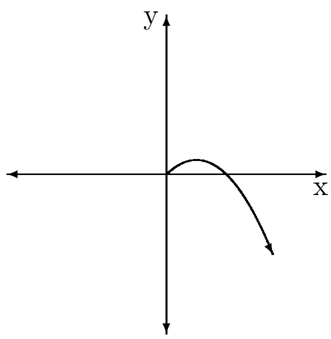
$$\begin{array}{ccc} f(1) = 1 & f(2) = 4 & f(a) = a^2 \\ f(-1) = 1 & f(-2) = 4 & f(-a) = a^2 \end{array}$$

An even function has a distinctive shape when graphed - the graph for the negative x's (the left-hand side of the y-axis) is a reflection of what is on the right-hand side of the y-axis.

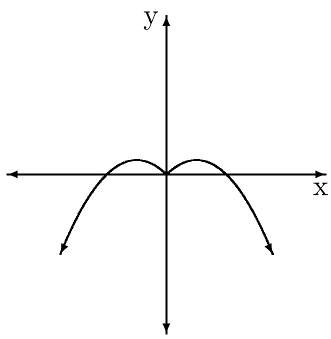


Each of the four graphs represents an even function.

Example 2 : Say we are given part of a graph of $g(x)$.



We now extend the graph of $g(x)$ to make an even function:



We now define what an odd function is. A function $f(x)$ is odd if

$$\boxed{f(-x) = -f(x)}$$

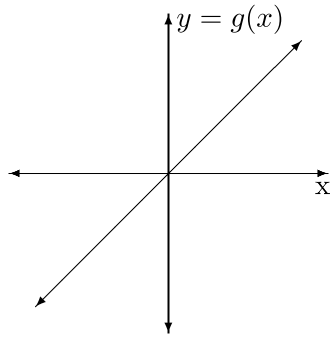
That is, if an input of a gives an output of b, then an input of $-a$ will give an output of $-b$.

Example 3 : The function $g(x) = x$ is an odd function. Note that

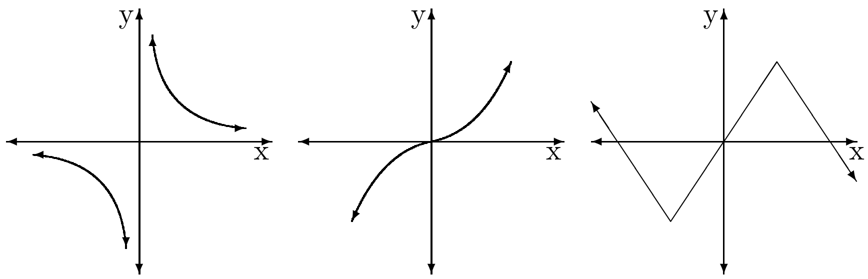
$$\begin{array}{cc} g(1) = 1 & g(2) = 2 \\ g(-1) = -1 & g(-2) = -2 \end{array}$$

The graph of an odd function has distinctive features; it has the property that you get what is on the right-hand side of the y-axis by rotating what is on the left-hand side of the y-axis through $180°$. And vice versa.

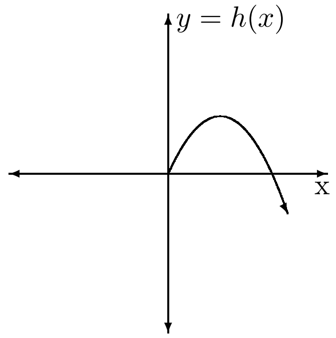
Example 4 :



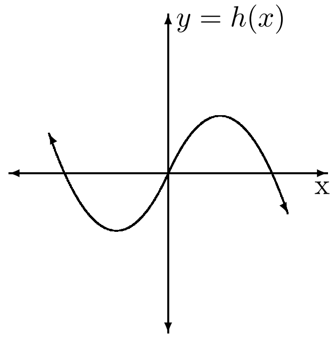
Here are more examples of odd functions.



Given the function $h(x)$:



We can extend $h(x)$ to be an odd function:



If we are given a function as an expression, we can test it to see if it is odd or even by substituting in a and $-a$ as inputs and finding out what the outputs are.

Example 5 : Is the function $y(x) = x^2 + 7$ even, odd, or neither?

$$\begin{aligned} y(a) &= a^2 + 7 \\ y(-a) &= (-a)^2 + 7 \\ &= a^2 + 7 \end{aligned}$$

We can see that $y(-a) = y(a)$, so $y(x)$ is even.

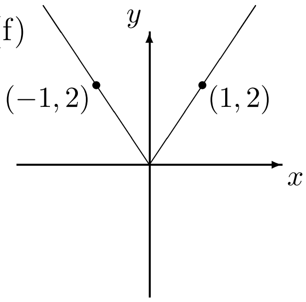
Exercises:

1. Are the following functions even, odd, or neither?

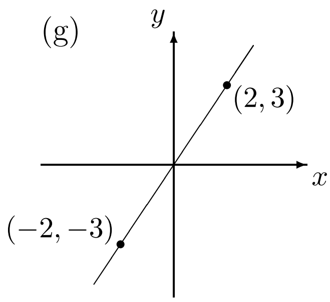
   (a) $f(x) = 2x$
   (b) $f(x) = x^2 - 2$
   (c) $f(x) = x + 1$
   (d) $f(x) = \frac{3}{x}$

(e) $f(x) = x^2 + x + 3$

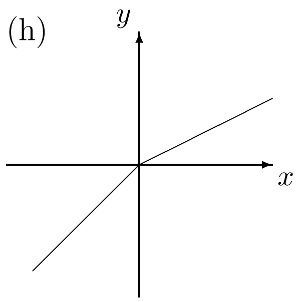
(f)



(g)



(h)



Section 4 The equation of a Circle

There are some equations that you should be able to recognize at first glance, and know roughly what they look like: the equation of a straight line is an example. Another equation that you should be able to recognize is the equation of a circle. It is

$$x^2 + y^2 = r^2$$

for some number r. When graphed, the set of points satisfying $x^2 + y^2 = r^2$ will be a circle of radius r centered at the origin. This means that the x and y-intercepts are $\pm r$.

The equation of a circle is not actually a function of x since each value of x has two possible values of y in the domain $-r < x < r$. But it is an equation that you will be expected to know how to graph.

A variation of the equation of a circle already given to you is

$$(x - a)^2 + (y - b)^2 = r^2$$

This is still an equation of a circle, but it is more general: it has a radius r as before, but is centred on (a, b) rather than at the origin.

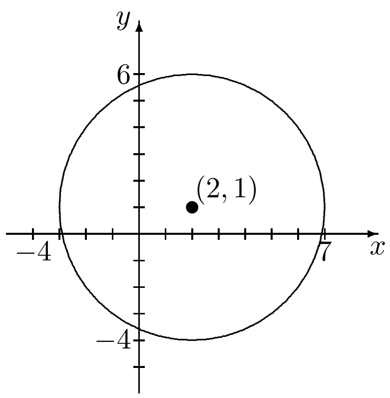
The circle

$$(x - 2)^2 + (y - 1)^2 = 25$$

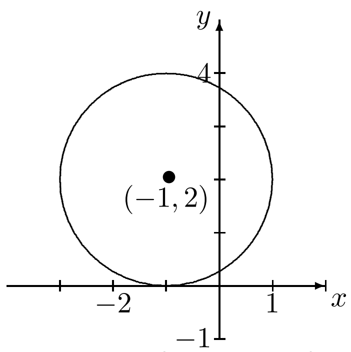
has centre $(2, 1)$ and radius 5. The circle

$$(x + 1)^2 + (y - 2)^2 = 4$$

has centre $(-1, 2)$ and radius 2.



$(x-2)^2+(y-1)^2=25$



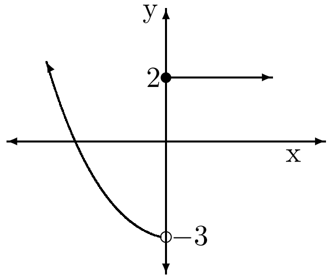
$(x+1)^2+(y-2)^2=4$

Exercises:

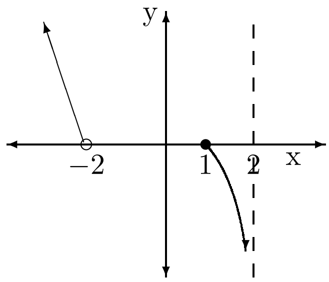
1. Write down the equation of a circle which has
   (a) radius 3 and centre $(6,-2)$
   (b) radius 5 and centre $(4,3)$
   (c) radius $1\frac{1}{2}$ and centre $(-1,2)$
   (d) radius 2 and centre $(\frac{1}{2},1\frac{1}{2})$
   (e) radius 4 and centre $(-1,-3)$

Exercise 3.2 Graphs

1. (a) What is the range of the function whose graph is below? Give your answer in interval notation.



   (b) What is the domain of the function whose graph is below (in interval notation)?



   (c) What are the x and y intercepts of $y = 2x - 3$?

   (d) The function $y = x^2 - 3x - 4$ crosses the x-axis twice and the y-axis once. Find all three intercepts.

   (e) Let $f(x) = x^3 - x$.

      i. Show that $f(x)$ is odd.

      ii. Sketch $f(x)$.

2. State whether the following equations represent a line, a parabola, a cubic, a circle, or a hyperbola.

   (a) $x^2 + y^2 = 16$

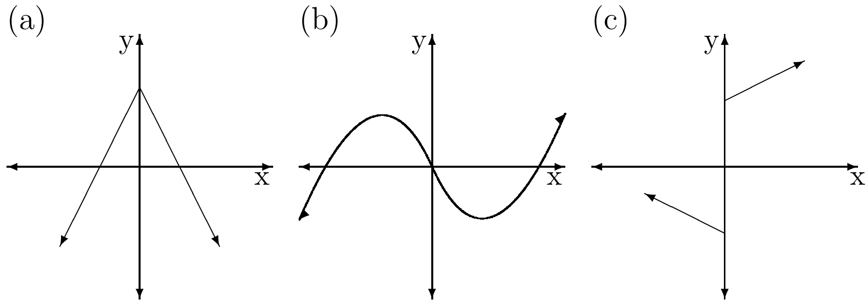
   (b) $y = x^2 + 5x + 6$

   (c) $y = \frac{2}{x}$

   (d) $(x - 1)^2 + (y + 2)^2 = 49$

   (e) $y = 2x + 1$

3. State whether the following functions are even, odd or neither:



Answers 3.2

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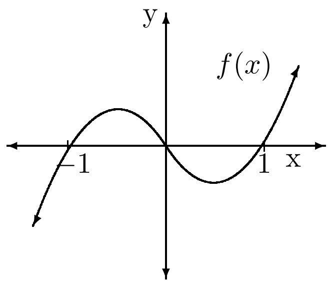
1. (a) $(-3, \infty)$
   (b) $(-\infty, -2)$ and $[1, 2)$
   (c) y-intercept -3; x-intercept $\frac{3}{2}$.
   (d) y-intercept -4; x-intercepts 4 and -1.
   (e)
   $$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -(x^3 - x) \\ &= -f(x) \end{aligned}$$
   so $f(x)$ is odd.



2. (a) Circle
   (b) Parabola
   (c) Hyperbola
   (d) Circle
   (e) Straight line
   (f) i. even
       ii. odd
       iii. neither