Worksheet 3.1 Functions

Section 1 Definitions

What is a function? A function can be thought of as a machine. It accepts an input, applies a rule to it and then produces an output. Diagrammatically, we might view the process like:

$$\text{input} \quad \rightarrow \quad \text{rule} \quad \rightarrow \quad \text{output}$$

> Example 1 : Rule f: take the input, and multiply it by 5. If we apply rule f to the input 4, we get $5 \times 4 = 20$.
>
> $$4 \quad \rightarrow \quad 5 \times 4 \quad \rightarrow \quad 20$$

What is the output when we apply rule f to the input x?

$$x \quad \rightarrow \quad 5 \times x \quad \rightarrow \quad 5x$$

As mentioned in other worksheets we look for shorthand ways of working with things. The shorthand way of writing "apply rule f to input 4" is to write $f(4)$. We say this as f of 4. So

$$f(4) = 20 \quad \text{and} \quad f(x) = 5x$$

We say the second item as f of x. When we apply rule f to input x our output gives us a shorthand way of writing the actual rule.

> Example 2 : We define rule G: take the input squared, and then add 5. Apply rule G to the inputs $-1, 1, a+1$ and x.
>
> $$\begin{aligned} G(-1) &= (-1)^2 + 5 = 6 \\ G(1) &= (1)^2 + 5 = 6 \\ G(a+1) &= (a+1)^2 + 5 = a^2 + 2a + 6 \\ G(x) &= x^2 + 1 \end{aligned}$$

There are several different ways of representing functions. The most common ways are

1. As a table of values
2. As a graph
3. As an algebraic expression

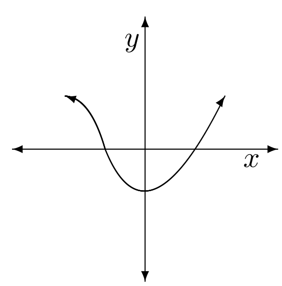
Here are some examples of the ways of representing a function.

Example 3 :

input	0	1	2	3	4	5	6
output	5	6	7	8	9	10	11

This is a table of values, and the rule isn't given explicitly in this case. However, we might be able to guess an appropriate rule. In this case it appears to be to take the input and add 5. With a table of values, the rule will not usually be given, and it may not be obvious what the rule is. But the table of values still represents a function. Let's give the rule that converts an input into an output a name, say f. Then the function that is associated with this table is $f(x) = x + 5$.

Example 4 :



This graph also represents a function, and from the graph we can learn things about the function. We will talk more about graphs in the next worksheet.

Example 5 : Consider

$$f(x) = 5x^2 + 2$$

Here we use the shorthand notation for the function rule, and the input is always some variable which is often x or t. The output is given by what the rule does to the variable, in this case x. Rule f in this case says take the input, square it, multiply the result by 5, and then add 2. This is the output.

Exercises:

1. For each part, find a function which describes the table of values

(a)

x	0	1	2	3
$f(x)$	-4	-3	-2	-1

(b)

x	2	4	5	6
$f(x)$	5	17	26	37

(c)

x	0	2	4	5
$f(x)$	1	5	9	11

2. Complete the table of values for the function $f(x) = 3 - x^2$.

x	-3	-1	2	3	5	7
$f(x)$						

Section 2 Substitution

When a function is represented algebraically, we are given the rule as it applies to some variable. This is called functional notation. To compute the rule applied to any input we simply replace the variable with the input.

Example 1 :

$$\begin{aligned} \text{Given: } f(x) &= x^2 \text{ then} \\ f(5) &= (5)^2 = 25 \\ f(-1) &= (-1)^2 = 1 \\ f(a+b) &= (a+b)^2 = a^2 + 2ab + b^2 \\ f(2y) &= (2y)^2 = 4y^2 \end{aligned}$$

Example 2 :

$$\begin{aligned} \text{Given: } g(t) &= 5t - 3 \text{ then} \\ g(1) &= 2 \\ g(0) &= -3 \\ g(10) &= 47 \\ g(x^2) &= 5x^2 - 3 \end{aligned}$$

Example 3 :

$$\begin{aligned} \text{Given: } h(x) &= \frac{1}{x} \text{ then} \\ h(1) &= \frac{1}{1} = 1 \\ h(3) &= \frac{1}{3} \end{aligned}$$

In example 3, $h(x) = \frac{1}{x}$. So $h(0)$ doesn't make sense since we can't divide by zero. When the function doesn't make sense for a particular input value, we say that the function is not defined for that input value.

Example 4 :

$$\begin{aligned} \text{Given: } f(x) &= 3x^2 + 2x + 2 \text{ then} \\ f(1) &= 3 \times (1)^2 + 2 \times 1 + 2 = 7 \\ f(0) &= 2 \\ f(-1) &= 3 \\ f(3y) &= 27y^2 + 6y + 2 \end{aligned}$$

Exercises:

1. Given $f(x) = 2x - x^2$, find
   (a) $f(3)$
   (b) $f(-2)$
   (c) $f(x-1)$
2. Given $f(x) = 2x^2 - x + 3$, find
   (a) $f(4)$
   (b) $f(-3)$
   (c) $f(x+2)$
3. Given $f(x) = (x+2)^2 - x + 3$, find
   (a) $f(2)$
   (b) $f(-2)$
   (c) $f(-4)$
   (d) $f(x+1)$

Section 3 Composition of Functions

This section deals with a thing called composition of functions. As a picture, composition looks like this:

$$\text{input}_1 \;\rightarrow\; \text{rule}_1 \;\rightarrow\; \text{output}_1 = \text{input}_2 \;\rightarrow\; \text{rule}_2 \;\rightarrow\; \text{output}_2$$

It is like having two machines, one after the other. The result from one machine forms the input to the next machine. We could also write it like this

$$x \;\rightarrow\; \text{rule } f \;\rightarrow\; f(x) \;\rightarrow\; \text{rule } g \;\rightarrow\; g(f(x))$$

So $g(f(x))$ is a composite function which may also be written $g \circ f(x)$. The circle may be taken to mean 'follows'.

Note: It is important to realize that

$$g \circ f(x) \neq f \circ g(x)$$

The order in which the functions are applied is important. It is equally, if not more important, to realize that

$$f \circ g(x) \neq f(x) \times g(x)$$

Example 1 :

Take $g(x) = x^2$ and $f(x) = 1 + x$. Then

$$\begin{aligned}
g \circ f(1) &= g(f(1)) = g(2) = 4 \\
g \circ f(3) &= g(f(3)) = g(4) = 16 \\
f \circ g(1) &= f(g(1)) = f(1) = 2 \\
f \circ g(3) &= f(g(3)) = f(9) = 10 \\
g \circ f(x) &= g(f(x)) = g(x+1) = (x+1)^2 \\
f \circ g(x) &= f(g(x)) = f(x^2) = x^2 + 1
\end{aligned}$$

When dealing with compositions, if you write it all out longhand as in the above examples, you shouldn't get too confused. It's when you try and do too much in your head that you get the computation around the wrong way.

Exercises:

1. Given $f(x) = x + 2$ and $g(x) = 2x$, find

   (a) $f(3)$
   (b) $g(5)$
   (c) $g \circ f(3)$
   (d) $g \circ f(-1)$
   (e) $f \circ g(3)$
   (f) $f \circ g(-4)$

2. Given $f(x) = x^2 - 1$ and $g(x) = 3 - x$, find

   (a) $g \circ f(1)$ (b) $g \circ f(t)$ (c) $g \circ f(4)$ (d) $f \circ g(x+1)$ (e) $f \circ g(x+2)$

Section 4 Functions from words

Functions are useful for determining the answer to many problems that occur in real-life situations. You may be required to take a problem that is given to you in words and come up with the function describing the information given. There is no hard and fast method of dealing with problems, although there are some general hints.

1. Read the information carefully, and translate as much as possible into mathematical expressions.
2. Try out a few inputs to get a feel for the rule before writing it down with a variable.

Example 1 : You have to create a rectangular paddock and you only have 1000 metres of fencing. You can choose the width of the paddock. Find a function that takes as input the width of the paddock and gives as an output the area of the paddock enclosed.

The perimeter of the paddock is 1000m. If the length is l and the width w then the perimeter p in symbols is $p = 2l + 2w$. The area A is $A = lw$, which we will try to write in terms of w, rather than w and l. From $2l + 2w = 1000$, we see that

$$\begin{aligned} 2l &= 1000 - 2w \\ l &= \frac{1}{2}(1000 - 2w) \\ &= 500 - w \end{aligned}$$

Then the area is

$$\begin{aligned} A &= lw \\ &= (500 - w)w \\ &= 500w - w^2 \end{aligned}$$

We can create the following table:

w	l	A
300	$500 - 300 = 200$	300×200
200	$500 - 200 = 300$	200×300
100	$500 - 100 = 400$	100×400
x	$500 - x$	$x(500 - x)$

The required function is

$$A = f(x) = x(500 - x)$$

where A is an obvious symbol to represent area.

In later worksheets on differentiation we will learn a technique that will allow us to find the optimum width so that fencing of the paddock will yield the maximum area.

Example 2 : A ferry carries an average of 300 people a day. The fare is $ 1.20. The UTA research shows that 50 extra people will travel per day for every 10cent fare reduction. Work out the function that has the number of fare reductions as input, and as output the total amount of money collected by the UTA each day.

Reductions	Fare	Number of People	Money collected
0	1.20	300	300×1.20
2	1.00	$300 + 2 \times 50$	$(300 + 2 \times 50) \times 1.00$
12	0.00	$300 + 12 \times 50$	$(300 + 12 \times 50) \times 0.00$
x	$1.20 - 0.10x$	$300 + x \times 50$	$(300 + x \times 50) \times (1.20 - 0.10x)$

So the function in terms of the number of reductions is

$$(300 + x \times 50) \times (1.20 - 0.10x)$$

In the worksheet on differentiation we will learn how to maximize the money taken in by the UTA.

Exercises:

1. A truck weighs 1500 kg and it is to be loaded with cartons each weighing 5 kg. Work out the function which has the number of cartons as input and the total weight of the truck as output.

2. A photocopier service costs $240 plus 2.5 cents for every copy made. Work out the function which has the number of copies made as the input and the total cost as output.

Exercises for Worksheet 3.1

1. (a) A mother records the height of her son over the first 8 months, and measurements she made are shown in this table:

Input (age in months)	0	2	4	6	8
Output (height in cm)	50	54	58	62	66

Express the output as a function of the input.

(b) Consider the pattern of triangles shown:

If the input is the number of horizontal rows in the pyramid, and the output is the number of triangles, describe the relationship between the input and the output.

(c) Evaluate x, given that $x = 2a^3 - 3\sqrt{a}$, when $a = 2.73$.

(d) Evaluate a, given that $v = \sqrt{u^2 + 2as}$, when $v = 10$ m/s, $u = 9$ m/s, and $s = 2$ m.

(e) The formula for converting degrees Fahrenheit (F) to Celsius (C) is given by $C = \frac{5}{9}(F - 32)$. Evaluate C when $F = 100$.

(f) If $f(x) = 3x^2 + 1$, find $f(-2)$.

(g) If $g(x) = \frac{3x+2}{x^2-1}$, find $g(\frac{1}{2})$.

(h) If $f(x) = x^2 - 2x + 3$, find $f(x + h)$.

(i) If $f(x) = 3x^2 - 2x + 4$, find $\frac{f(x+h)-f(x)}{h}$.

(j) For $f(x)$ given in the previous question, evaluate $\frac{f(x+h)-f(x)}{h}$ when $x = 2$ and $h = 0.001$.

2. (a) If $f(x) = x^2$ and $g(x) = \frac{1}{x}$, find

   i. $f \circ g(x)$

   ii. $g \circ f(x)$.

(b) If $f(x) = 3x^2$ and $g(x) = x - 3$, find

   i. $f \circ g(x)$

   ii. $g \circ f(x)$.

(c) If $f(x) = x^2$, $g(x) = x + 1$, and $h(x) = 2x$, find $f \circ g \circ h(x)$.

(d) If $f(x) = x^2$, find

   i. $f(2)$

   ii. $f(x + h)$

   iii. $f(2x)$

iv. $f(x+1)$.

(e) If $f(x) = \frac{1}{2x+1}$, find

i. $f(\frac{1}{2})$

ii. $f(3+x)$

iii. $f(x^2)$.

3. (a) Here is a rule for a function: take the input, multiply it by 3, then add 4, then square the result.

i. Express the output as a function of the input.

ii. Evaluate the output when the input equals -2.

(b) Here is another rule for a function: take the input, subtract 2, take the square root of the result, then add 5.

i. Express the output as a function of the input.

ii. Evaluate the output when the input equals 2.

(c) There are 27 times as many cars as motorcycles in Australia. If C represents the number of cars, and M the number of motorcycles, write an equation describing the relationship between M and C.

Answers to Test Three
and
Exercises from Worksheets 3.1 - 3.10

Answers to Test Three

1. (a) $9x^2 + 2$
   (b) $(2x+1)^2$
2. (a) $[-4, 0] \cup (2, 4]$
   (b) $x = -4$
3. (a) $\frac{\pi}{3}$
   (b) $\phi = \frac{\pi}{6}$
4. (a) $-\frac{1}{\sqrt{2}}$
   (b) $\sqrt{3}$
5. (a) $k = 2$
   (b) $u = -4$, $y = -6$
6. (a) 47
   (b) 10
7. (a) 10
   (b) No
8. (a) $3x^2 + 6x$
   (b) $(0, 0)$, $(-2, 4)$
9. (a) Max
   (b) $x = -\frac{1}{3}$
10. (a) $5\cos(5x+2)$
    (b) $3e^x$

Worksheet 3.1

1. (a) Output $= 50 + 2\times$ Input
   (b) Output $=$ Input $\times$ Input
   (c) $x \doteq 35.74$
   (d) 4.75
   (e) $37\frac{7}{9}$
   (f) 13
   (g) $-4\frac{2}{3}$
   (h) $(x+h)^2 - 2(x+h) + 3$
   (i) $6x - 2 + 3h$
   (j) 10.003
2. (a) i. $\frac{1}{x^2}$
   ii. $\frac{1}{x^2}$
   (b) i. $3(x-3)^2$
   ii. $3x^2 - 3$
   (c) $(2x+1)^2$
   (d) i. $f(2) = 4$
   ii. $f(x+h) = (x+h)^2$
   iii. $f(2x) = 4x^2$
   iv. $f(x+1) = (x+1)^2$
   (e) i. $f(\frac{1}{2}) = \frac{1}{2}$
   ii. $f(3+x) = \frac{1}{2x+7}$
   iii. $f(x^2) = \frac{1}{2x^2+1}$

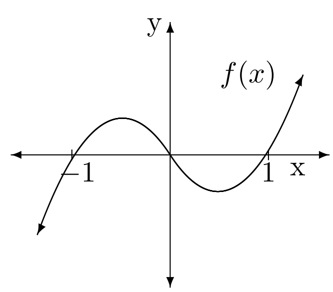
3. (a) i. If the input is x then the output is $(3x+4)^2$
   ii. 4
   (b) i. If the input is x, then the output is $\sqrt{x-2}+5$
   ii. 5
   (c) $C = 27M$

<u>Worksheet 3.2</u>

1. (a) $(-3, \infty)$
   (b) $(-\infty, -2)$ and $[1, 2)$
   (c) y-intercept -3; x-intercept $\frac{3}{2}$.
   (d) y-intercept -4; x-intercepts 4 and -1.
   (e)

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -(x^3 - x) \\ &= -f(x) \end{aligned}$$

so $f(x)$ is odd.



2. (a) Circle
   (b) Parabola
   (c) Hyperbola
   (d) Circle
   (e) Striaght line
   (f) i. even
   ii. odd
   iii. neither

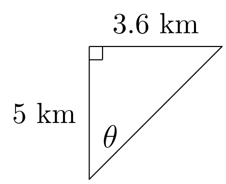
Worksheet 3.3

1. (a) i. $150°$ ii. $240°$ iii. $\frac{4\pi}{9}$ iv. $\frac{7\pi}{30} = 0.733$

   (b) i. $\frac{1}{\sqrt{2}}$ ii. $\frac{\sqrt{3}}{2}$ iii. $\sqrt{3}$

2. (a) $\sqrt{108}$ (b) $\frac{\pi}{3}$

3. (a)



   (b) $35.75°$

Worksheet 3.4

1. (a) $\frac{1}{\sqrt{2}}$
   (b) $\frac{1}{\sqrt{3}}$
   (c) $\frac{1}{\sqrt{2}}$
   (d) $-\frac{1}{2}$
   (e) $\frac{1}{2}$
   (f) $-\sqrt{3}$

2. (a) $\frac{\sqrt{3}+1}{2\sqrt{2}}$
   (b) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

3. (a) $\frac{\pi}{3}$, $\frac{4\pi}{3}$
   (b) $\frac{\pi}{4}$, $\frac{3\pi}{4}$
   (c) $\frac{3\pi}{4}$, $\frac{5\pi}{4}$

Worksheet 3.5

1. (a) None
   (b) One
   (c) Infinite
   (d) One
   (e) None

2. (a) $x = 8$, $y = 13$
   (b) $x = \frac{11}{5}, y = \frac{2}{5}$
   (c) $x = \frac{1}{17}$, $y = -\frac{5}{17}$

3. Peter is 15, Anneka is 9

Worksheet 3.6

1. (a) Arithmetic (b) Geometric (c) Neither (d) Neither (e) Arithmetic

2. (a) $T_6 = 15$, $T_{20} = 99$, $S_{10} = 120$
   (b) $T_6 = \log 7 + 5 \log 2$, $T_{20} = \log 7 + 19 \log 2$, $S_{10} = \frac{10}{2}(2 \log 7 + 9 \log 2)$
   (c) $T_6 = 2$, $T_{20} = 2^{15}$, $S_{10} = \frac{1}{16}(2^{10} - 1)$
   (d) $T_6 = (0.5)(0.9)^5$, $T_{20} = (0.5)(0.9)^{19}$, $S_{10} = 5(1 - .9^{10})$
   (e) $T_6 = -2$, $T_{20} = -\frac{1}{2^{13}}$, $S_{10} = \frac{128}{3}(1 + 2^{-10})$

3. (a) $a = 506$, $d = -18$ (b) $81(1 - (\frac{1}{3})^5)$ (c) $T_2 = (\frac{9}{4})^3$ (d) $n^2 + 6n$

4. (a) $437,988.84 (b) 5 metres

Worksheet 3.7

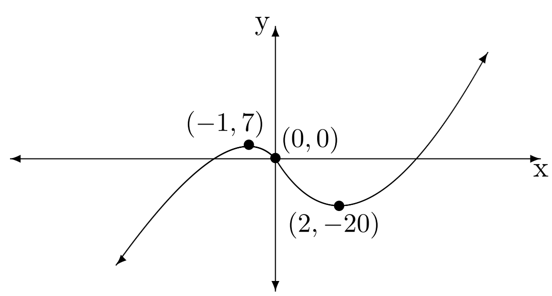
1. (a) 10
   (b) 6
   (c) $\frac{1}{3}$
   (d) $\frac{25}{29}$
   (e) 0

2. (a) Continuous
   (b) Continuous
   (c) Continuous
   (d) Not continuous
   (e) Not continuous

Worksheet 3.8

2. (a) $2x + 6$
   (b) $21x^2 - 10x + 9$
   (c) $\frac{1}{2\sqrt{x}} + 8$
   (d) $-6x^{-3} - x^{-2}$
   (e) $-\frac{2}{x^3} - \frac{1}{x^2} + 6$

3. (a)

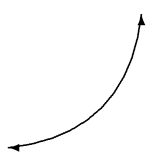


(b) $T'(s) = -3$. The temperature is dropping 3 degrees for every km above sea level.

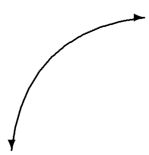
Worksheet 3.9

1. (a) $\Delta > 0$, 2 roots
   (b) $\Delta < 0$, no roots
   (c) $\Delta > 0$, 2 roots
   (d) $\Delta = 0$, 1 root
   (e) $\Delta < 0$, no roots

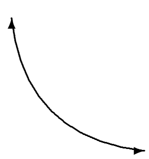
2. (a) Concave up and increasing.



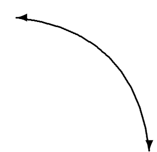
(b) Concave down and increasing.



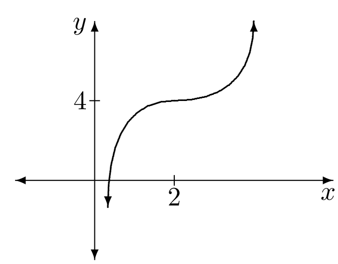
(c) Concave up and decreasing.



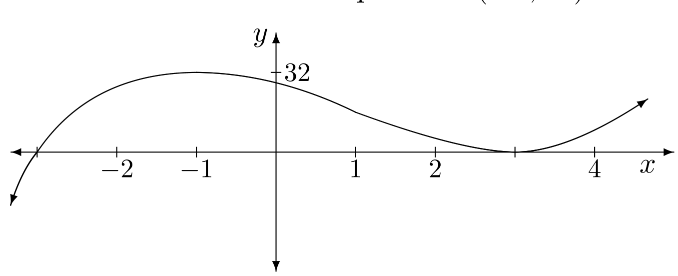
(d) Concave down and decreasing.



(e) Change of concavity at $(2, 4)$.



3. (a) $f(x) = (x-3)^2(x+3)$. Intercepts at $x = 3, -3$. $f'(x) = 3(x-3)(x+1)$. Stationary points at $x = 3, -1$. $f''(x) = 6x - 6$. $f''(3) > 0$ so there is a minimum point at $(3, 0)$. $f''(-1) < 0$ so there is a maximum point at $(-1, 32)$.



(b) $\frac{dD}{dt} > 0$ and $\frac{d^2D}{dt^2} < 0$

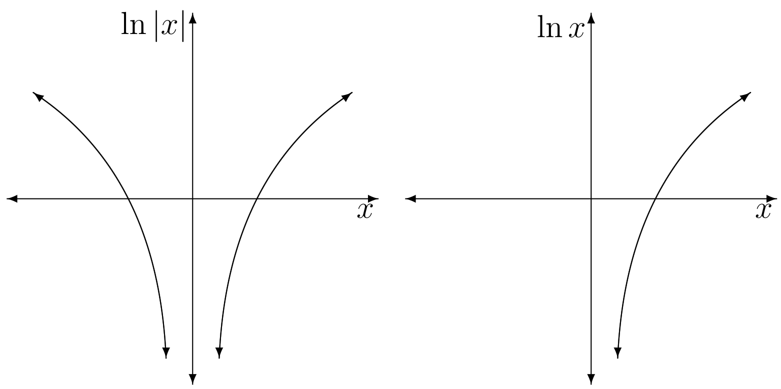
Worksheet 3.10

1. (a) $\frac{1}{x}$
   (b) $\frac{2}{2x-3}$
   (c) $\frac{2x-3}{x^2-3x+2}$
   (d) $4xe^{x^2}$
   (e) $-\frac{1}{3}(2x+2)e^{x^2+2x-1}$
   (f) $3\cos 3x$
   (g) $-\frac{3}{2}\sin\frac{x}{2}$
   (h) $x\mathrm{sec}^2 x^2$
   (i) $\frac{3}{x} - 6\cos 3x$
   (j) $\sin 2x + \frac{6x}{3x^2-1}$

2. (a)

x	3.0	2.0	1.0	0.1	0.0	-0.1	-1.0	-2.0
$\ln x$	1.098	0.69	0	-2.3	-	-	-	-
$\ln \lvert x\rvert$	1.098	0.69	0	-2.3	-	-2.3	0	0.69

   (b)



   (c) $g(x) = \ln|x|$ is defined on the interval $(-\infty, 0)$ and $(0, \infty)$, while $h(x) = \ln x$ is defined on $(0, \infty)$ only. Since $f(x) = \frac{1}{x}$ is defined on the same domain as $g(x)$, then $g(x)$ is a better representation of the integral than $h(x)$.

3. (a) i. -1
      ii. $y = -x$
      iii. $y = x$
   (b) i. $\frac{\ln 2}{0.02} \approx 35$
      ii. $\frac{dP}{dt} = 0.082e^{0.02t}$. This represents the rate of change of population per year (in billions of people per year).
      $\left.\frac{dP}{dt}\right]_{t=0} = 0.082$
      In 1975 the population was increasing at a rate of 82 million people per year.
      $\left.\frac{dP}{dt}\right]_{t=15} = 0.1107$
      In 1990 the population was increasing at a rate of 110.7 million people per year.