Worksheet 2.9 Introduction to the Cartesian Plane

Section 1 The Cartesian Plane

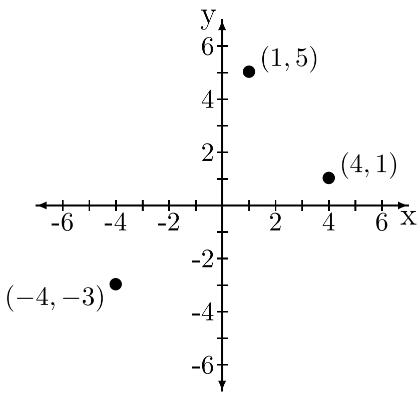
In worksheet 2.4 we discussed inequalities on the number line. This is a type of graph in one dimension. For many maths problems, we need to draw graphs in two dimensions. Graphs contain a lot of information at a glance, and so are a very useful tool.

A graph in two dimensions will arise from an equation with two variables. For example, $x + y = 0$. This equation has two variables, x and y, and there is a relationship between them which the formula expresses. To represent this equation as a graph we draw a picture of all the ordered pairs (x, y) which satisfy the relationship. The picture is placed on a cartesian plane, which is the two dimensional equivalent of the number line, and is formed by placing two number lines at right angles to each other intersecting at $(0, 0)$. This point is called the origin.

The entries in an ordered pair (x, y) are called the coordinates. In this pair, x is the first coordinate (or independent variable), and y is the second coordinate (or dependent variable). The horizontal number line represents the first coordinate of the ordered pair and the vertical line the second co-ordinate. You should label each number line (called axis) with the variable that it corresponds to.

To draw the picture, we first need to find some of the ordered pairs (x, y) which satisfy the equation. This is usually done by putting in values of x and finding the corresponding values of y. Once we have some ordered pairs, we plot them on the cartesian plane. First find the x-value, then move up or down that line to find the necessary y-value. You may well have done a similar thing when looking up a street in a street directory. Each map in a directory is a cartesian plane with letters on the horizontal axis and numbers on the vertical axis. To find $G8$ on a certain page, you find G, and then move up or down the G column until you get to the line marked 8.

Example 1 : We will plot the following ordered pairs: $(1, 5)$, $(-4, -3)$, and $(4, 1)$.



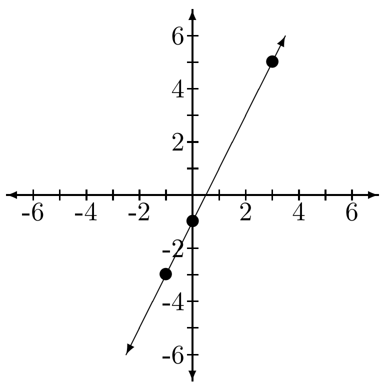
Example 2 : Plot the line $y = 2x - 1$.

We need to find some ordered pairs that satisfy the equation.

When $x = 0$, $y = -1$, so $(0, -1)$ is an ordered pair.
When $x = 3$, $y = 5$, so $(3, 5)$ is an ordered pair.
When $x = -1$, $y = -3$, so $(-1, -3)$ is an ordered pair.

We could construct a table to represent this information:

x	0	-1	3
y	-1	-3	5



Exercises:

1. Complete the table of values for the given equations.

   (a) $y = x - 3$

x	0	1	2
y			

   (b) $2x + y = 4$

x	0		
y		2	6

   (c) $y = 5 - x$

x		2	
y	6		1

2. Complete the table of values and graph the ordered pairs on a number plane:

   (a) $y = x - 3$

x	0	1	2
y			

   (b) $y = x + 1$

x	0	1	2
y			

   (c) $y = 6 - 2x$

x	0	1	2
y			

Section 2 Introduction to a line

The equation $ax + dy + c = 0$ represents a straight line on a cartesian plane, where a, d, and c may be any numbers. Any equation that can be put in a form that looks like this is a straight line. The important thing to notice about this equation is that the x and y variables are taken to the first power and the first power only, and they are not multiplied together; that is, there are no terms like xy, x^2, y^2, $\sqrt{x}$ or $\frac{1}{x}$.

Example 1 : Is $x + y = 0$ a straight line? Yes: we would have $a = 1$, $d = 1$, and $c = 0$.

Example 2 : Is $3x^2 + y = 2$ a straight line? No: the x variable is squared - raised to the power of 2 - so this equation is not of the required form.

Example 3 : Is $x = 5$ a straight line? Yes: we would have $a = 1$, $d = 0$, and $c = -5$.

Example 4 : Is $y = 5x$ a straight line? Yes: we would have $a = -5$, $d = 1$, and $c = 0$.

Exercises:

1. Which of the following equations represents a straight line?
   (a) $x - 2y - 6 = 0$
   (b) $xy - 8 = 0$
   (c) $\frac{1}{x} + y - 6 = 0$
   (d) $y = 2x - 1$
   (e) $x(x + y) = 4$

Section 3 Slope-Intercept Form

There is another way to write the equation of a line which allows more information about the graphed line to be readily available. We rearrange the formula $ax + dy + c = 0$ in the following way:

$$\begin{aligned} ax + dy + c &= 0 \\ ax + dy &= -c \\ dy &= -ax - c \\ y &= -\frac{a}{d}x - \frac{c}{d} \end{aligned}$$

Now since a, b and c are constants then so is $-\frac{a}{d}$ a constant together with $-\frac{c}{d}$. Let's rename these fractions by writing

$$m = -\frac{a}{d} \quad \text{and} \quad b = -\frac{c}{d}$$

Then our equation is written

$$y = mx + b$$

This form is what we will call the slope-intercept form of the straight-line equation. The slope of a line is also called the gradient, or sometimes it is referred to as rise over run. The rise of a line is the number of units up (or down) for every so many units that we move along the line. So, if a line has a slope of 2, then it rises 2 units for every unit that it goes across. If a line has a slope of $\frac{1}{2}$, then it rises 1 unit for every 2 units that it goes across. If the equation of the line is written in the form $y = mx + b$, then m is the slope, or gradient, of the line, and b is called the y-intercept. This is the value that y has when $x = 0$.

Note: Sometimes either m or b (or both!) might be equal to zero. Such an expression still represents a straight line.

What are the gradients and y-intercepts of the following straight lines?

Example 1 : $y = 2x + 3$. The gradient is 2, and the y-intercept 3.

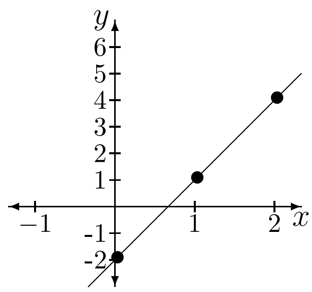
Example 2 : $y = \frac{5}{2}x + 3$. The gradient is $\frac{5}{2}$, and the y-intercept 3.

Example 3 : $y = 2 - 5x$. The gradient is -5, and the y-intercept 2.

Example 4 : $2y = 2x + 2$. The gradient is 1, and the y-intercept 1.

Let us graph the equation $y = 3x - 2$. Make a table of values for a selection of x-values, then plot the points onto a diagram, then connect them with a straight line:

x	0	1	2
y	-2	1	4

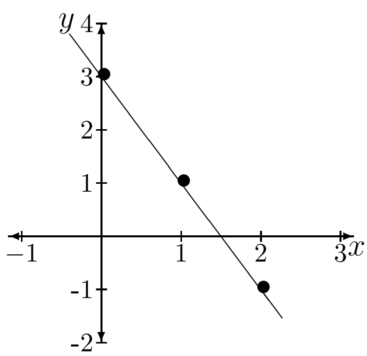


The gradient is defined as the rise over the run. Consider the rise over the run as we go from the point $(1, 1)$ to the point $(2, 4)$. The rise is $4 - 1 = 3$, and the run is $2 - 1 = 1$. Therefore the gradient is

$$\begin{aligned} \text{gradient} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

Let us graph the equation $y = -2x + 3$. Make a table of values for a selection of x-values, then plot the points onto a diagram, then connect them with a straight line:

x	0	1	2
y	3	1	-1



The gradient is defined as the rise over the run. Consider the rise over the run as we go from the point $(0, 3)$ to the point $(1, 1)$. The rise is $1 - 3 = -2$, and the run is $1 - 0 = 1$. Therefore the gradient is

$$\begin{aligned} \text{gradient} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{-2}{1} \\ &= -2 \end{aligned}$$

Two lines are parallel if they have the same gradient. If two lines are parallel, and they also have the same y-intercept, then they coincide (in other words, they are the same line). Parallel lines with different y-intercepts will never intersect.

If two lines do not have the same gradient, they will eventually meet at some point. In other words, for some x value, they will both have the same y value. This is called the point of intersection of the lines. Two lines are perpendicular if the product of the gradients of the two lines is -1 (we will not prove this). Let two lines be

$$y = m_1 x + b_1 \quad \text{and} \quad y = m_2 x + b_2$$

The two lines are perpendicular if $m_1 m_2 = -1$.

Example 5 : The lines

$$\begin{aligned} y &= 5x + 2 \\ y &= 5x + 3 \end{aligned}$$

are parallel, and never meet. They both have a gradient of 5.

Example 6 : The lines

$$\begin{aligned} y &= 5x + 4 \\ 2y &= 10x + 8 \end{aligned}$$

coincide. The second equation could be divided by 2 to become identical with the first equation.

Example 7 : The lines

$$\begin{aligned} y &= \frac{1}{5}x + 3 \\ y &= -5x + 2 \end{aligned}$$

are perpendicular, since $\frac{1}{5} \times -5 = -1$.

Example 8 : The lines

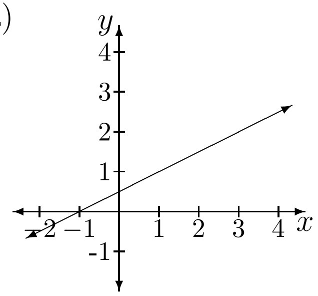
$$y = 10x + 3$$
$$y = 5x + 10$$

meet when $x = \frac{7}{5}$ and $y = 17$. We will discuss how to find out where two lines meet in a later worksheet.

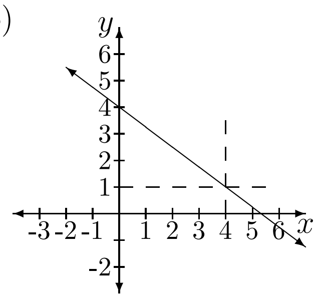
Exercises:

1. Find the gradient and y-intercept of each of the following straight lines and hence write down the equation of the straight line.

(a)



(b)



2. Write the following equations in the form $y = mx + b$, and hence find the gradient (m) and and the y-intercept (b).

(a) $y = 2x + 1$

(b) $x + y + 3 = 0$

(c) $2y = 8x - 6$

(d) $x - y + 8 = 0$

(e) $2x + 3y + 12 = 0$

(f) $\frac{1}{3}x - \frac{1}{2}y = \frac{1}{6}$

Section 4 A line through two points

Given any two points, we can find the equation of the straight line that joins them. To do so, we first work out the gradient. Recall from an earlier section that we talked about the gradient as the rise of a line divided by its run. We use this to find the gradient. The rise of a line is the difference between two y-values. The run is the difference in the corresponding x-values. The ratio of these gives the gradient. Say we are given two points on a line: (x_1, y_1) and (x_2, y_2). The slope of the line is then

$$\text{Slope } = m = \frac{y_2 - y_1}{x_2 - x_1}$$

It makes no difference which point you take as your first, but whichever one you use as your first y-value, you must use the corresponding x as your first x-value. Once you have the gradient you then use it in another rise over run ratio, except this time with just one of the points given - either point will do - and with arbitrary x and y values. So the two points we are now considering are, say, (x_2, y_2) and (x, y) and the slope m (which we now know) is given by

$$m = \frac{y - y_2}{x - x_2}$$

which we can rearrange to put it in the slope-intercept form:

$$\begin{aligned} y - y_2 &= m(x - x_2) \\ y &= mx - mx_2 + y_2 \\ y &= mx + b \end{aligned}$$

where we have let $b = -mx_2 + y_2$.

Example 1 : What is the equation of the line that $(5, 2)$ and $(4, 6)$ lie on?

$$m = \frac{2 - 6}{5 - 4} = -4$$

Now that we have found the gradient, all we have to do is find the y-intercept. Choose either of the two points that we know lie on the line. We will choose $(4, 6)$. Now substitute in values for x, y and m to find b, the y-intercept.

$$\begin{aligned} y &= mx + b \\ 6 &= (-4)4 + b \\ b &= 22 \end{aligned}$$

so the equation of the line joining $(5, 2)$ and $(4, 6)$ is $y = 22 - 4x$.

Example 2 : What is the equation of the line that $(1, 3)$ and $(2, 3)$ lie on?

$$m = \frac{3 - 3}{1 - 2} = \frac{0}{-1} = 0$$

so the gradient is 0. Now we again substitute in values of x, y and m to find b. We find that $y = 3$, so the equation of the line joining $(1, 3)$ and $(2, 3)$ is $y = (0)x + 3$ or simply $y = 3$.

Example 3 : What is the equation of the line that $(1, 5)$ and $(1, 2)$ lie on?

$$m = \frac{5 - 2}{1 - 1} = \frac{3}{0}$$

which doesn't make sense as we can't divide by zero.

If this happens, it means that the line is vertical. We can think of it as having an infinite slope. A vertical line through a point (a, b) has the equation $x = a$. So, for this example, the equation of the line is $x = 1$.

Example 4 : What is the equation of the line that $(3, 2)$ and $(-5, 2)$ lie on? The gradient is given by

$$m = \frac{2-2}{3-(-5)} = \frac{0}{8} = 0$$

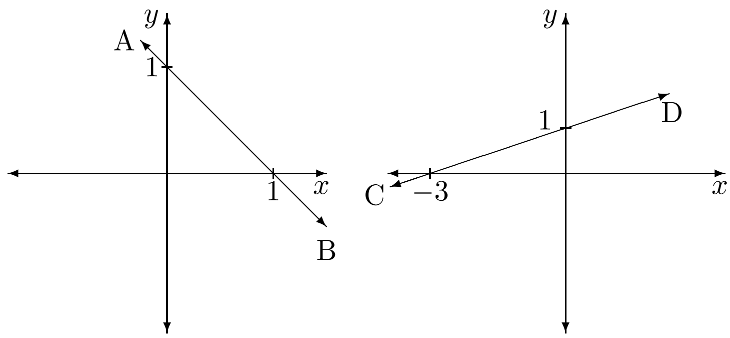
Since the slope is zero, and the line passes through a point whose y-value is 2, the equation of the line is $y = 2$.

Exercises:

1. Find the equation of the straight line passing through:
   - (a) $(0, -1)$ with gradient $\frac{3}{2}$
   - (b) $(2, 4)$ and $(3, 1)$
   - (c) $(0, 3)$ and $(2, 7)$
   - (d) $(1, 4)$ and $(3, 4)$
   - (e) $(3, 1)$ and is parallel to $y = 3x - 7$
   - (f) $(12, -2)$ and is perpendicular to $y = 3x - 7$

Exercises 2.9 Introduction to the Cartesian Plane

1. (a) What are the gradients of the lines AB and CD in Figure 1?



(b) Plot the points $P(-2,3)$, $Q(1,8)$, $R(-4,2)$, and $S(0,7)$. Is PQ parallel to RS? What are the slopes of PQ and RS?

(c) Complete the table of values for the equation $y = 3x + 2$, and use the table to help you draw the graph of $y = 3x + 2$.

x	-1	0	2	
y				0

Where does the graph intersect the x-axis?
Where does the graph intersect the y-axis?

(d) i. Is $y = 3x + y^2$ a straight line? Give reasons.
ii. Is $y + 15 = 0$ a straight line? Give reasons.

(e) What is the gradient and y-intercept of:

i. $y = -2x - 7$

ii. $2y - 8x - 1 = 0$

(f) What is the gradient of the line that is perpendicular to $y = 5x - 2$?

(g) Which of the following points lies on the line $y = 2x + 12$?

i. $(-20, 23)$

ii. $(\frac{1}{2}, 13)$

2. (a) Plot the points $X(-1, 2)$ and $Y(1, 3)$.
    i. What is the equation of the straight line that joins them?
    ii. What is the equation of the straight line which passes through X, but is perpendicular to XY?
    iii. What is the equation of the vertical line through Y?
    iv. What is the equation of the line through Y and also parallel to $y = 3x + 12$?

   (b) Find the equation of the line with gradient 2 that passes through the point $(0, -6)$.

   (c) Find the line parallel to $y = 1.5x + 4$ which goes through $(2, 4)$.

   (d) Find the line parallel to the x-axis which goes through $(22, 27)$.

   (e) What is the equation of the line that passes through the points $(-1, -15)$ and $(-10, -1)$?

3. (a) Plot the points $A(2, 4)$, $B(1, 2)$, and $C(3, 2)$. Find the point D that would make $ABCD$ a parallelogram.

   (b) A line has gradient 1, and goes through $(5, 2)$. Does the point $(10, 7)$ lie on the line?

   (c) A line has gradient $-L$. Could both $(2, 5)$ and $(0, -10)$ be on the line?

Answers 2.9

Section 1

1. (a)

x	0	1	2
y	-3	-2	-1

(b)

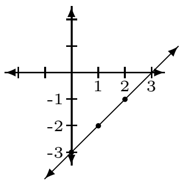
x	0	1	-1
y	4	2	6

(c)

x	-1	2	4
y	6	3	1

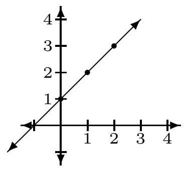
2. (a)

x	0	1	2
y	-3	-2	-1



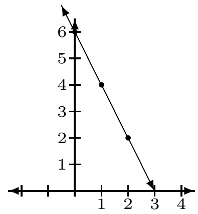
(b)

x	0	1	2
y	1	2	3



(c)

x	0	1	2
y	6	4	2



Section 2

1. (a) & (d)

Section 3

1. (a) $m = \frac{1}{2}$, $b = \frac{1}{2}$
   $y = \frac{1}{2}x + \frac{1}{2}$

   (b) $m = -\frac{3}{4}$, $b = 4$
   $y = -\frac{3}{4}x + 4$

2. (a) $m = 2$, $b = 1$
   (b) $m = -1$, $b = -3$
   (c) $m = 4$, $b = 3$
   (d) $m = 1$, $b = 8$
   (e) $m = -\frac{2}{3}$, $b = -4$
   (f) $m = \frac{2}{3}$, $b = -\frac{1}{3}$

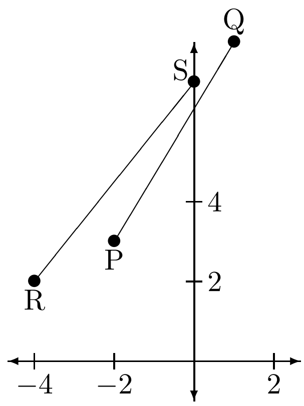
Section 4

1. (a) $y = \frac{3}{2}x - 1$
   (b) $y = -3x + 10$
   (c) $y = 2x + 3$
   (d) $y = 4$
   (e) $y = 3x - 8$
   (f) $y = -\frac{1}{3}x + 2$

Exercises 2.9

1. (a) i. Line AB has slope -1 ii. Line CD has slope $\frac{1}{3}$

(b) Line PQ has slope $1\frac{2}{3}$
Line RS has slope $1\frac{1}{4}$
PQ and RS are not parallel



(c)

x	-1	0	2	$-2/3$
y	-1	2	8	0

x-intercept $-\frac{2}{3}$ y-intercept 2

(d) i. No ii. Yes

(e) i. $m = -2$, y intercept -7 ii. $m = 4$, y intercept $\frac{1}{2}$

(f) $-\frac{1}{5}$

(g) i. No ii. Yes

2. (a) i. $y = 0.5x + 2.5$
ii. $y = -2x$
iii. $x = 1$
iv. $y = 3x$

(b) $y = 2x - 6$

(c) $y = 1.5x + 1$

(d) $y = 27$

(e) $y = \frac{-14}{9}x - \frac{149}{9}$

3. (a) $D(4, 4)$ There are other choices.
(b) $(10, 7)$ is on the line
(c) Only for $L = -\frac{15}{2}$