Worksheet 2.8 Introduction to Trigonometry

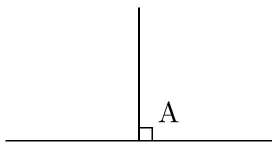
Section 1 Introduction to Plane Geometry

When two lines cross each other to make four angles all exactly the same, we call these two lines perpendicular. The angle between each of the lines is 90 deg or $\frac{\pi}{2}$ radians. We also can call an angle of $\frac{\pi}{2}$ radians a right angle, and it would be indicated on a picture by a small square in the angle.

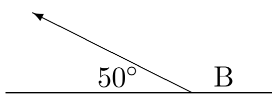
Two lines are parallel if it is possible to draw a line which is perpendicular to both lines. An arrow sitting on both lines in a diagram indicates that they are parallel.

A line forms what is called a straight angle. It is the same as if we were facing one direction, and then did an about face. We have moved through 180 deg, or π radians. To change directions on a line we must do the same thing: move through 180 deg. When we do a full turn or revolution on the plane, we move through 360 deg or 2π radians. So a straight angle is π radians and a full turn is 2π radians.

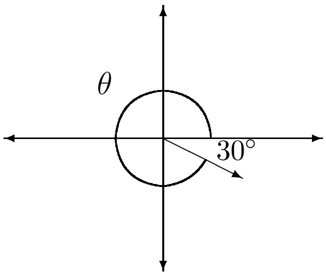
Example 1 : Angle A is a right angle.



Example 2 : Angle B is given by $B = 180° - 50° = 130°$.

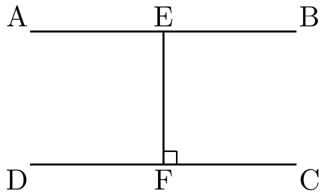


Example 3 : What is θ?



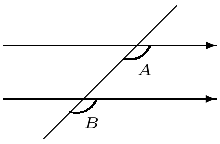
$$\theta = 360° - 30°$$

Example 4 : If CD and AB are parallel, and EFC is a right angle, what is FEB?

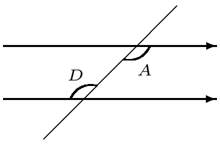


EFC is a right angle also.

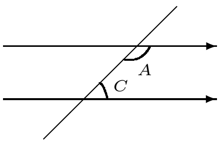
When pairs of parallel lines are both cut by another straight line, the various angles formed have some properties.



Angles A and B are called corresponding angles and are equal.

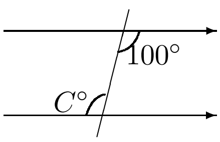


A and D are called alternate angles and are also equal.



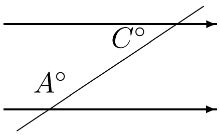
A and C are called co-interior angles and are supplementary, which means that $A + C = 180°$.

Example 5 : What is angle C?



C is 100°.

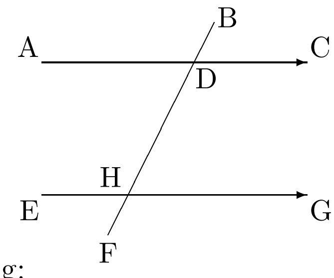
Example 6 : What is angle A, if $C^\circ = 45^\circ$?



$$\begin{aligned} A &= 180^\circ - 45^\circ \\ &= 135^\circ \end{aligned}$$

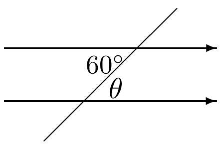
Exercises:

1. Name the following pairs of angles:
   (a) $\angle ADF$ and $\angle DHG$
   (b) $\angle BDC$ and $\angle DHG$
   (c) $\angle ADH$ and $\angle DHE$
   (d) $\angle BDA$ and $\angle DHE$

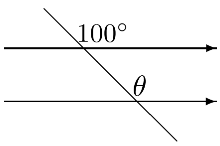


2. Find the value of θ in each of the following:

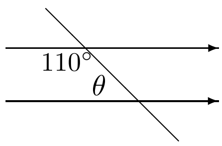
(a)



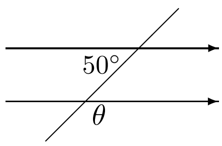
(b)



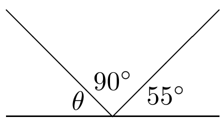
(c)



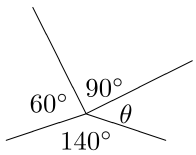
(d)



(e)



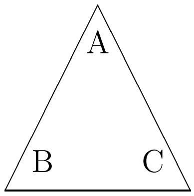
(f)



Section 2 Triangles

A triangle has three sides, which are normally denoted with lower-case letters and the opposite angle denoted by the corresponding capital letter. A triangle is often described by its three vertices. The interior angles of a triangle add up to $180\,\text{deg} = \pi$ radians. Thus, given two angles in a triangle, we can work out the third angle. If one of the angles in a triangle is a right angle, it is a right-angled triangle.

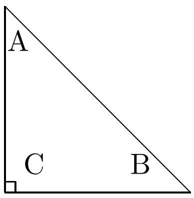
Example 1 : If angle A is 60°, and angle B is also 60°, what is angle C?



$$C = 180^\circ - 60^\circ - 60^\circ = 120^\circ - 60^\circ = 60^\circ$$

The triangle ABC in the above example is a special triangle called an equilateral triangle. Any triangle with three equal angles is an equilateral triangle. All the angles must be 60°. The sides, also, have an equal length.

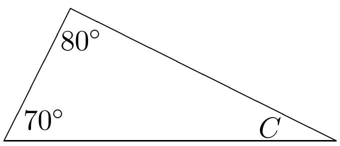
Example 2 : If angle A is 45°, what is angle B?



Since C is a right angle, we know that $C = 90^\circ$. Therefore $B = 180^\circ - 90^\circ - 45^\circ = 45^\circ$.

There is another special triangle called the isosceles triangle; it has two angles the same. The lengths of the sides opposite the equal angles are the same. Isosceles triangles can come in various shapes as the third angle can vary. The third angle doesn't have to be a right angle.

Example 3 : What is the angle C?



$$C = 180^\circ - 80^\circ - 70^\circ = 30^\circ$$

You will have noticed that, when referring to angles, we have often given two units of measure: degrees and radians - the most important unit being the radian. If the unit of measure is not specified on a given angle, the angle is assumed to be in radians. To convert degrees to radians, we use the following:

$$\pi \text{ radians } = 180^\circ$$

Therefore $1^\circ = \frac{\pi}{180}$ radians. So

$$60^\circ = 60 \times \frac{\pi}{180} = \frac{\pi}{3} \text{ radians}$$

A similar argument is used to convert radians to degrees.

$$\begin{aligned} \pi \text{ radians} &= 180^\circ \\ 1 \text{ radian} &= \frac{180^\circ}{\pi} \end{aligned}$$

Therefore, if we want to find out how many degrees there are in $\frac{\pi}{4}$ radians, we calculate as follows:

$$\frac{\pi}{4} \text{ radians} = \frac{\pi}{4} \times \frac{180^\circ}{\pi} = 45^\circ$$

Exercises:

1. Convert the following angles in degrees to radians; write the answers in terms of π.

   (a) 60° (d) 45° (g) 240°
   (b) 90° (e) 100° (h) 80°
   (c) 120° (f) 360° (i) 300°

2. Convert the following angles in degrees to radians; write the answers to 2 decimal places.

   (a) 30° (d) 100° (g) 240°
   (b) 40° (e) 45° (h) 600°
   (c) 90° (f) 160° (i) 300°

3. Convert the following angles in radians to angles in degrees:

   (a) $\frac{\pi}{2}$ (d) 2.5 (g) $\frac{2\pi}{3}$
   (b) $\frac{\pi}{6}$ (e) $\frac{\pi}{7}$ (h) $\frac{3\pi}{2}$
   (c) $\frac{\pi}{8}$ (f) $\frac{\pi}{12}$ (i) $\frac{5\pi}{6}$

Section 3 Pythagoras' Theorem

All right-angled triangles obey the theorem of Pythagoras, which states:

> The square of the length of the hypotenuse is equal to the sum of the squares of the other two sides.

Construct two identical squares, and divide the sides into lengths a and b as shown.

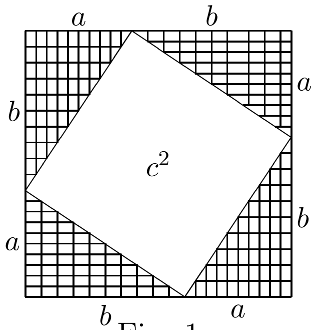


Fig. 1

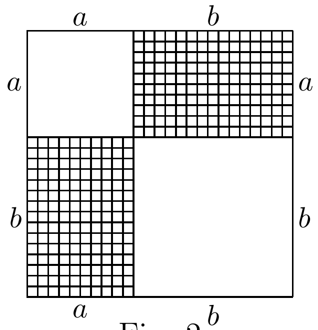
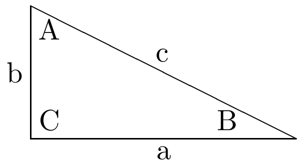


Fig. 2

$$\begin{aligned}
\text{Shaded area in Fig. 1} &= 4 \times \frac{1}{2}ab = 2ab \\
\text{Shaded area in Fig. 2} &= 2 \times ab = 2ab \\
c^2 + 2ab &= (a+b)^2 \\
&= a^2 + 2ab + b^2 \\
\text{therefore } a^2 + b^2 &= c^2
\end{aligned}$$

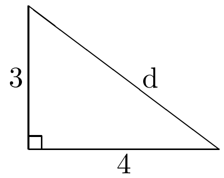
The hypotenuse is the side opposite the right angle. We now put this information onto a diagram and into a formula.



ABC is a right angled triangle, and c is the hypotenuse. Then a, b, and c satisfy the relationship:

$$a^2 + b^2 = c^2$$

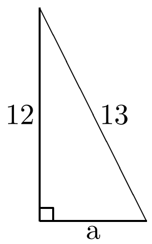
Example 4 : What is d?



$$\begin{aligned} d^2 &= 3^2 + 4^2 \\ d^2 &= 25 \\ d &= 5 \end{aligned}$$

Note that, since d is a length, we take the positive solution of the quadratic equation $d^2 = 25$.

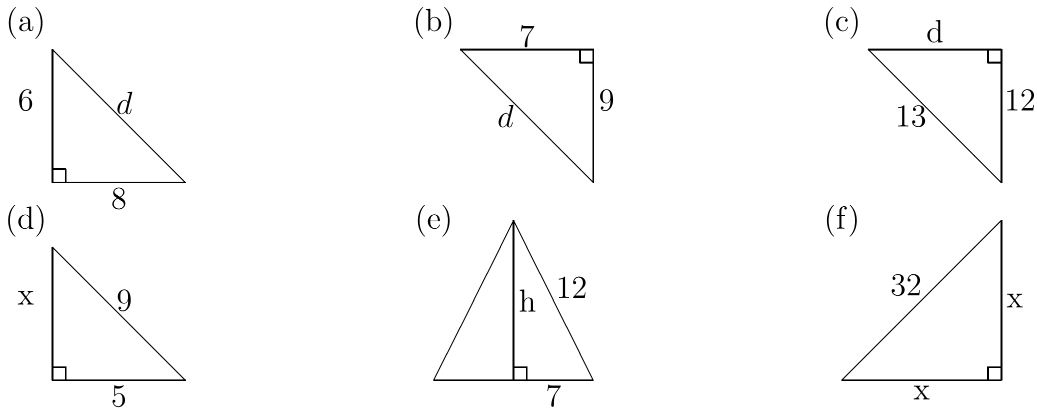
Example 5 : What is a?



$$\begin{aligned} 13^2 &= a^2 + 12^2 \\ 169 &= a^2 + 144 \\ 169 - 144 &= a^2 \\ 25 &= a^2 \\ 5 &= a \end{aligned}$$

Exercises:

1. Use Pythagoras' theorem to find the length of the unknown side:

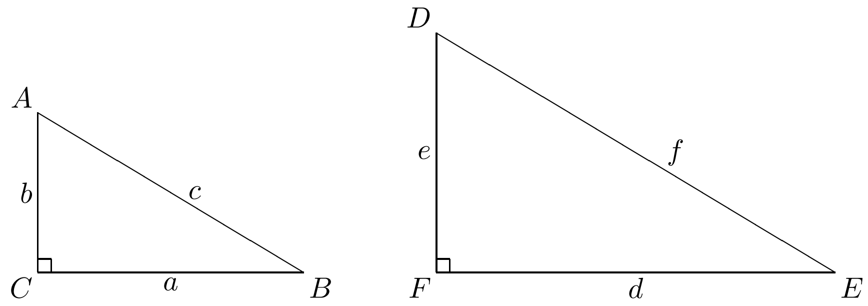


2. (a) A 4m long ladder is leaning against a wall. The foot of the ladder is 1.2 meters out from the wall. How far up the wall does the ladder reach? Draw a diagram!

   (b) In a triangle ABC, $\angle ABC = 90°$, $AC = 20$cm, and $AB = 9$ cm. Find the length of BC.

Section 4 Introductory Trigonometry

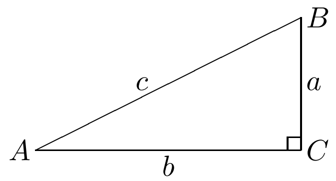
Similar triangles are ones which have the same shape. All the internal angles are the same. Similar triangles may be different sizes. The triangles ABC and DEF drawn below are similar but not the same.



The length of the sides of these triangles are proportional. That is, we can find one number p that gives

$$\begin{aligned} e &= pb \\ f &= pc \\ d &= pa \end{aligned}$$

In addition, corresponding angles are the *same*. Since the triangles formed by certain angles are proportional, we can describe angles by looking at the ratios of the sides around them. For the moment, we will restrict our discussion to that of right-angled triangles. Consider

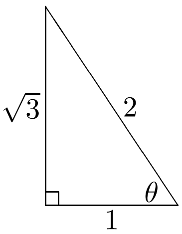


Recall that the side opposite the right angle is called the hypotenuse. In this case it is c. The side opposite the angle in question (in this case A) is called the opposite side. The remaining side is called the adjacent side, in this case b.

The trigonometric ratios are the ratios of various pairs of sides. They are sine (usually written sin), cosine (usually written cos), and tangent (usually written tan). The definitions of these ratios, in terms of right angled triangles are:

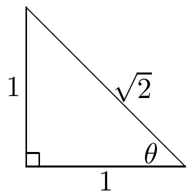
$$\begin{aligned}
\sin A &= \frac{\text{opposite side}}{\text{hypotenuse}} \\
&= \frac{a}{c} \\
\cos A &= \frac{\text{adjacent side}}{\text{hypotenuse}} \\
&= \frac{b}{c} \\
\tan A &= \frac{\text{opposite side}}{\text{adjacent side}} \\
&= \frac{a}{b}
\end{aligned}$$

Example 1 :



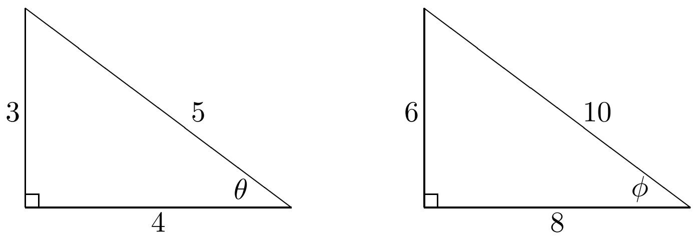
$$\begin{aligned}
\sin\theta &= \frac{\text{OPP}}{\text{HYP}} = \frac{\sqrt{3}}{2} \\
\cos\theta &= \frac{\text{ADJ}}{\text{HYP}} = \frac{1}{2} \\
\tan\theta &= \frac{\text{OPP}}{\text{ADJ}} = \frac{\sqrt{3}}{1}
\end{aligned}$$

Example 2 :



$$\begin{aligned}
\sin\theta &= \frac{1}{\sqrt{2}} \\
\cos\theta &= \frac{1}{\sqrt{2}} \\
\tan\theta &= 1
\end{aligned}$$

Example 3 : How are θ and ϕ related?



Since $\sin\theta = \frac{3}{5}$ and $\sin\phi = \frac{6}{10} = \frac{3}{5}$, θ and ϕ must be the same angle.

Exercises:

1. Find the following ratios:

   (a) $\sin\theta$

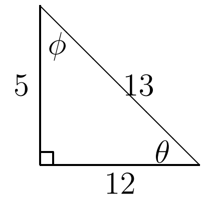
   (b) $\cos\theta$

   (c) $\tan\theta$

   (d) $\sin\phi$

   (e) $\cos\phi$

   (f) $\tan\phi$



2. Find the following ratios:

   (a) $\sin\theta$

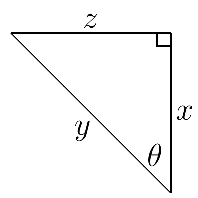
   (b) $\cos\theta$

   (c) $\tan\theta$

   (d) $\sin(90-\theta)$

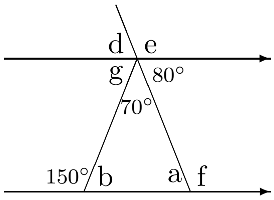
   (e) $\cos(90-\theta)$

   (f) $\tan(90-\theta)$



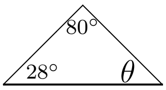
Exercises 2.8 Introduction to Trigonometry

1. Find the value of each of the pronumerals:

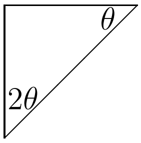


2. Find the value of θ in each of the following:

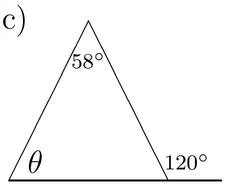
(a)



(b)



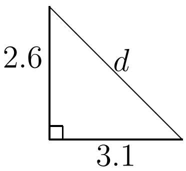
(c)



3. (a) Change the angle $\frac{7\pi}{3}$ to degrees.
   (b) Change the angle $\frac{6\pi}{5}$ to degrees.
   (c) Change 87° to radians. Write the answer to 2 decimal places.

4. Find the value of each pronumeral:

(a)



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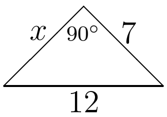

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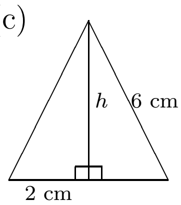
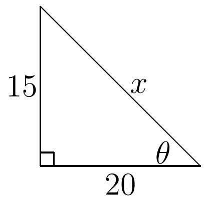


For part (c), assume the largest triangle is isosceles.

5. (a) Find the value of x
   (b) Find
      i. $\cos\theta$
      ii. $\sin\theta$
      iii. $\tan\theta$



Answers 2.8

Section 1

1. (a) alternate (b) corresponding (c) co-interior (d) corresponding

2. (a) 60° (b) 100° (c) 70° (d) 130° (e) 35° (f) 70°

Section 2

1. (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$ (e) $\frac{5\pi}{9}$ (f) 2π (g) $\frac{4\pi}{3}$ (h) $\frac{4\pi}{9}$ (i) $\frac{5\pi}{3}$

2. (a) 0.52 (b) 0.70 (c) 1.57 (d) 1.75 (e) 0.79 (f) 2.79 (g) 4.19 (h) 10.47 (i) 5.24

3. (a) 90° (b) 30° (c) 22.5° (d) 143° (e) 26° (f) 15° (g) 120° (h) 270° (i) 150°

Section 3

1. (a) 10 (b) 11.40 (c) 5 (d) 7.48 (e) 9.75 (f) 22.63

2. (a) 3.816 metres (b) 17.86 cm

Section 4

1. (a) $\frac{5}{13}$ (b) $\frac{12}{13}$ (c) $\frac{5}{12}$ (d) $\frac{12}{13}$ (e) $\frac{5}{13}$ (f) $\frac{12}{5}$

2. (a) $\frac{z}{y}$ (b) $\frac{x}{y}$ (c) $\frac{z}{x}$ (d) $\frac{x}{y}$ (e) $\frac{z}{y}$ (f) $\frac{x}{z}$

Exercises 2.8

1. $a = 80°$, $b = 30°$, $d = 80°$, $e = 100°$, $f = 100°$, $g = 30°$

2. (a) $72°$ (b) $30°$ (c) $62°$

3. (a) $420°$ (b) $216°$ (c) 1.52

4. (a) 4.05 (b) 9.75 (c) 5.66

5. (a) 25

   (b) i. $\frac{4}{5}$ ii. $\frac{3}{5}$ iii. $\frac{3}{4}$