# Worksheet 2.5 Arithmetic with Surds

## Section 1 Expanding Expressions Involving Surds

Fractional powers and the basic operations on them are introduced in worksheet 1.8. This worksheet expands on the material in that worksheet and also on the material introduced in worksheet 1.10. The distributive laws discussed in worksheet 1.10 are

$$\begin{aligned} a(b+c) &= ab+ac \\ (b+c)d &= bd+cd \end{aligned}$$

You may also recall that we showed the expansion of

$$(a+b)(x+y) = a(x+y) + b(x+y) = ax + ay + bx + by$$

We will now use these to expand expressions involving surds.

Example 1 :

$$\begin{aligned} (1+\sqrt{3})(1+\sqrt{3}) &= 1(1+\sqrt{3}) + \sqrt{3}(1+\sqrt{3}) \\ &= 1 + \sqrt{3} + \sqrt{3} + \sqrt{3} \times \sqrt{3} \\ &= 1 + 2\sqrt{3} + 3 \\ &= 4 + 2\sqrt{3} \end{aligned}$$

Note: You should recall from worksheet 1.8 that

$$\sqrt{3} \times \sqrt{3} = 3$$

and more generally,

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

Example 2 :

$$\begin{aligned} (1+\sqrt{5})(2+\sqrt{3}) &= 1(2+\sqrt{3}) + \sqrt{5}(2+\sqrt{3}) \\ &= 2 + \sqrt{3} + 2\sqrt{5} + \sqrt{15} \end{aligned}$$

Example 3 :

$$\begin{aligned} \sqrt{2}(5+\sqrt{8}) &= 5\sqrt{2} + \sqrt{16} \\ &= 5\sqrt{2} + 4 \end{aligned}$$

Note: After expansion of this expression we ended up with a perfect square inside a square root sign. This was simplified. In a similar way surds that have perfect squares as factors should be simplified as far as possible. For example,

$$\begin{aligned}
\sqrt{20} &= \sqrt{4}\times\sqrt{5} = 2\sqrt{5}\\
\sqrt{75} &= \sqrt{25}\times\sqrt{3} = 5\sqrt{3}\\
\sqrt{32} &= \sqrt{16}\times\sqrt{2} = 4\sqrt{2}
\end{aligned}$$

Exercises:

1. Simplify the following surds:

   (a) $\sqrt{12}$ (b) $\sqrt{125}$ (c) $\sqrt{48}$ (d) $\sqrt{72}$ (e) $\sqrt{27}$

2. Expand and simplify the following expressions:

   (a) $\sqrt{2}(3+\sqrt{5})$
   (b) $\sqrt{6}(\sqrt{2}+\sqrt{8})$
   (c) $4(\sqrt{5}+3)$
   (d) $(2+\sqrt{3})(1+\sqrt{3})$
   (e) $(3-\sqrt{5})(3-2\sqrt{5})$
   (f) $(5-\sqrt{2})(5+\sqrt{2})$
   (g) $(2+\sqrt{5})(2+\sqrt{3})$
   (h) $(1-\sqrt{2})(1+\sqrt{3})$
   (i) $(8-\sqrt{2})(8+\sqrt{2})$
   (j) $(\sqrt{3}+\sqrt{5})(\sqrt{3}+\sqrt{5})$

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## Section 2 Fractions involving surds

As in the last worksheet on algebraic fractions, fractions involving surds are worked out similarly to fractions involving numbers. When adding and subtracting fractions the denominators must be the same for all the fractions involved in the calculation. This will normally involve finding equivalent fractions with the right denominator.

Example 1 :

$$\begin{aligned}
\frac{2}{\sqrt{2}+\sqrt{8}}+\frac{1}{3} &= \frac{3\times 2}{3(\sqrt{2}+\sqrt{8})}+\frac{\sqrt{2}+\sqrt{8}}{3(\sqrt{2}+\sqrt{8})}\\
&= \frac{6+\sqrt{2}+\sqrt{8}}{3(\sqrt{2}+\sqrt{8})}
\end{aligned}$$

Being able to factorize expressions involving surds often makes the expression much tidier. For this example we can get

$$\begin{aligned}
\frac{6+\sqrt{2}+\sqrt{8}}{3(\sqrt{2}+\sqrt{8})} &= \frac{6+\sqrt{2}(1+\sqrt{4})}{3\sqrt{2}(1+\sqrt{4})} \\
&= \frac{6+3\sqrt{2}}{9\sqrt{2}} \\
&= \frac{3(2+\sqrt{2})}{9\sqrt{2}} \\
&= \frac{2+\sqrt{2}}{3\sqrt{2}} \\
&= \frac{\sqrt{2}(\sqrt{2}+1)}{3\sqrt{2}} \\
&= \frac{\sqrt{2}+1}{3}
\end{aligned}$$

which is much tidier than the previous expression. We could have saved time and effort by simplifying the expression first. We needed to note that 8 has a factor which is a perfect square so

$$\sqrt{8} = \sqrt{2} \times \sqrt{4} = 2\sqrt{2}$$

and our initial sum becomes

$$\begin{aligned}
\frac{2}{\sqrt{2}+\sqrt{8}} + \frac{1}{3} &= \frac{2}{\sqrt{2}+2\sqrt{2}} + \frac{1}{3} \\
&= \frac{2}{3\sqrt{2}} + \frac{1}{3} \\
&= \frac{2}{3\sqrt{2}} + \frac{\sqrt{2}}{3\sqrt{2}} \\
&= \frac{2+\sqrt{2}}{3\sqrt{2}} \\
&= \frac{\sqrt{2}+1}{3}
\end{aligned}$$

Example 2 :

$$
\begin{aligned}
\frac{\sqrt{7}+1}{3}+\frac{\sqrt{7}-2}{4} &= \frac{4(\sqrt{7}+1)}{3\times 4}+\frac{3(\sqrt{7}-2)}{4\times 3}\\
&= \frac{4\sqrt{7}+4}{12}+\frac{3\sqrt{7}-6}{12}\\
&= \frac{4\sqrt{7}+4+3\sqrt{7}-6}{12}\\
&= \frac{7\sqrt{7}-2}{12}
\end{aligned}
$$

Example 3 :

$$
\begin{aligned}
\frac{\sqrt{2}-3}{\sqrt{5}}-\frac{\sqrt{2}+4}{2} &= \frac{2(\sqrt{2}-3)}{2\sqrt{5}}-\frac{\sqrt{5}(\sqrt{2}+4)}{2\sqrt{5}}\\
&= \frac{2\sqrt{2}-6-\sqrt{10}-4\sqrt{5}}{2\sqrt{5}}
\end{aligned}
$$

The last expression could be manipulated a little more, if desired, by noting that $\sqrt{10}=\sqrt{2}\sqrt{5}$.

Exercises:

1. Simplify the following:

(a) $\frac{\sqrt{2}}{4}+\frac{\sqrt{2}+1}{5}$

(b) $\frac{\sqrt{3}+2}{5}+\frac{\sqrt{3}+5}{7}$

(c) $\frac{\sqrt{6}-1}{\sqrt{3}}+\frac{\sqrt{6}+2}{2\sqrt{3}}$

(d) $\frac{\sqrt{5}-8}{4\sqrt{2}}-\frac{\sqrt{5}-3}{5\sqrt{2}}$

(e) $\frac{4}{\sqrt{2}+1}+\frac{3}{\sqrt{2}+5}$

(f) $\frac{4}{\sqrt{3}+\sqrt{7}}+\frac{5}{\sqrt{3}+2\sqrt{7}}$

(g) $\frac{8+\sqrt{5}}{2}-\frac{4-2\sqrt{5}}{3}$

(h) $\frac{2(\sqrt{5}+1)}{6}-\frac{3(\sqrt{5}+3)}{21}$

(i) $\frac{4}{\sqrt{2}-1}+\frac{4+\sqrt{3}}{\sqrt{2}-3}$

(j) $\frac{3(\sqrt{2}-7)}{\sqrt{5}}-\frac{5(\sqrt{2}+1)}{2}$

## Section 3 Rationalizing the denominator

'Rationalizing the denominator' means to get all the fractional powers out of the denominator of a fraction. After rationalizing there should only be whole numbers on the bottom of the fraction and no surds. In effect what we want to do is find an equivalent fraction. You already know that to find an equivalent fraction you need to multiply the top and bottom of the fraction by the same number or expression, which effectively multiplies by 1. Therefore to rationalize the denominator we need to find an expression which, when multiplied with an expression containing surds, gives only fractions or whole numbers. To achieve this, it is important to notice that

$$(a-b)(a+b) = a(a+b) - b(a+b) = a^2 - b^2$$

Note: $a^2 - b^2$ is called the difference of squares.

The importance of the difference of squares formula is that if $a$ and $b$ are both surds (for example $a = \sqrt{2}$, $b = \sqrt{3}$), then the expression $a^2 - b^2$ contains *no* surds.

Example 1 :

$$\begin{aligned} (1+\sqrt{2})(1-\sqrt{2}) &= 1(1-\sqrt{2}) + \sqrt{2}(1-\sqrt{2}) \\ &= 1 - \sqrt{2} + \sqrt{2} - \sqrt{2} \times \sqrt{2} \\ &= 1 - 2 \\ &= -1 \end{aligned}$$

Example 2 :

$$\begin{aligned} (3+\sqrt{5})(3-\sqrt{5}) &= 3(3-\sqrt{5}) + \sqrt{5}(3-\sqrt{5}) \\ &= 9 - 3\sqrt{5} + 3\sqrt{5} - 5 \\ &= 4 \end{aligned}$$

We can use this information to help us rationalize the denominator of fractions with expressions containing square roots in the denominator.

Example 3 :

$$\frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Example 4 :

$$\begin{aligned}\frac{5}{3\sqrt{7}} &= \frac{5}{3\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} \\ &= \frac{5\sqrt{7}}{21}\end{aligned}$$

Example 5 :

$$\begin{aligned}\frac{6}{5\sqrt{2}} &= \frac{6}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{6\sqrt{2}}{10} \\ &= \frac{3\sqrt{2}}{5}\end{aligned}$$

Example 6 :

$$\begin{aligned}\frac{\sqrt{3}+1}{\sqrt{3}-4} &= \frac{\sqrt{3}+1}{\sqrt{3}-4} \times \frac{\sqrt{3}+4}{\sqrt{3}+4} \\ &= \frac{\sqrt{3}(\sqrt{3}+4)+1(\sqrt{3}+4)}{3-16} \\ &= \frac{3+4\sqrt{3}+\sqrt{3}+4}{-13} \\ &= \frac{7+5\sqrt{3}}{-13} \\ &= -\frac{7+5\sqrt{3}}{13}\end{aligned}$$

Example 7 :

$$\begin{aligned}\frac{1}{1+\sqrt{3}} &= \frac{1}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}} \\ &= \frac{1-\sqrt{3}}{1-3} \\ &= \frac{\sqrt{3}-1}{2}\end{aligned}$$

Example 8 :

$$\begin{aligned}\frac{1}{3-\sqrt{5}} &= \frac{1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}} \\ &= \frac{3+\sqrt{5}}{9-5} \\ &= \frac{3+\sqrt{5}}{4}\end{aligned}$$

Example 9 :

$$\begin{aligned}\frac{\sqrt{2}}{6+\sqrt{2}} &= \frac{\sqrt{2}}{6+\sqrt{2}} \times \frac{6-\sqrt{2}}{6-\sqrt{2}} \\ &= \frac{\sqrt{2}(6-\sqrt{2})}{36-2} \\ &= \frac{6\sqrt{2}-2}{34} \\ &= \frac{3\sqrt{2}-1}{17}\end{aligned}$$

Exercises:

1. Rewrite the following expressions with rational denominators:

   (a) $\frac{3}{\sqrt{5}}$

   (b) $\frac{4}{\sqrt{8}}$

   (c) $\frac{9}{\sqrt{48}}$

   (d) $\frac{\sqrt{2}+1}{\sqrt{2}}$

   (e) $\frac{\sqrt{3}-1}{\sqrt{5}}$

   (f) $-\frac{4}{3\sqrt{2}}$

   (g) $\frac{\sqrt{5}+3}{\sqrt{10}}$

   (h) $\frac{\sqrt{2}-1}{\sqrt{7}}$

   (i) $\frac{1}{\sqrt{3}-1}$

   (j) $\frac{4}{\sqrt{6}-2}$

   (k) $\frac{7}{\sqrt{7}-2}$

   (l) $\frac{-3}{\sqrt{5}+1}$

   (m) $\frac{\sqrt{2}+3}{\sqrt{5}}$

   (n) $\frac{\sqrt{5}-1}{\sqrt{5}+3}$

   (o) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+4}$

   (p) $\frac{5+2\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

# Exercises 2.5 Arithmetic with Surds

1. (a) $\sqrt{2} \times \sqrt{2}$
   (b) $(\sqrt{5})^2$
   (c) $\sqrt{3} \times \sqrt{2}$
   (d) $6\sqrt{3} + 2\sqrt{3}$
   (e) $2(3 - \sqrt{7})$
   (f) $(2 + \sqrt{2}) - (3 + 2\sqrt{2})$
   (g) $3\sqrt{3} - 2(\sqrt{3} + 2)$
   (h) $\sqrt{8}$
   (i) $\sqrt{a^2}$
   (j) $\sqrt{49b^4}$
   (k) $\frac{1}{p}\sqrt{\frac{p^4}{16}}$
   (l) $5\sqrt{\frac{25}{4}}$

2. Simplify:
   (a) $5\sqrt{12} + \sqrt{27}$
   (b) $3\sqrt{20} - 3\sqrt{80} - 2\sqrt{45}$
   (c) $\sqrt{15} \times \sqrt{3}$
   (d) $2\sqrt{20} \times \frac{\sqrt{45}}{3}$
   (e) $(2 + \sqrt{3})(2 - \sqrt{3})$
   (f) $(3\sqrt{2} + 2\sqrt{3})^2$
   (g) $(3\sqrt{5} - 2)(\sqrt{5} + 3)$
   (h) $(2\sqrt{5} - \sqrt{3})(3\sqrt{3} + \sqrt{5})$
   (i) $\frac{2}{\sqrt{3}}$
   (j) $\frac{2}{\sqrt{5}+1}$
   (k) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$
   (l) $\frac{5}{\sqrt{2}+\sqrt{8}} - \frac{1}{4}$
   (m) $\frac{2}{\sqrt{3}+1} + \frac{4}{\sqrt{3}-2}$
   (n) $\frac{1}{\sqrt{2}+3} - \frac{3}{\sqrt{2}-1}$

3. (a) Find the value of $x^2 + 4x + 4$ when $x = 2 + \sqrt{3}$.
   (b) Find the value of $2x^2 - 3xy$ when $x = \sqrt{2} + 3$ and $y = \sqrt{2} - 2$.
   (c) Given $\sqrt{2} = 1.41$ (to 2 decimal places), simplify $\frac{1}{\sqrt{2}}$ without a calculator. (Rationalize the denominator.)
   (d) Given $\sqrt{3} = 1.73$ (to 2 decimal places), simplify $\frac{3}{2+\sqrt{3}}$ without a calculator.
   (e) Find the area of a circle of radius $2\sqrt{7}$ cm (correct to 2 decimal places).
   (f) Find the perimeter of a rectangle of length $(3 + \sqrt{2})$ and breadth $(\sqrt{2} - 1)$ cm.

## Answers 2.5

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Section 1

1. (a) $2\sqrt{3}$ (b) $5\sqrt{5}$ (c) $4\sqrt{3}$ (d) $6\sqrt{2}$ (e) $3\sqrt{3}$

2. (a) $3\sqrt{2}+\sqrt{10}$
   (b) $6\sqrt{3}$
   (c) $4\sqrt{5}+12$
   (d) $5+3\sqrt{3}$
   (e) $19-9\sqrt{5}$
   (f) $23$
   (g) $4+2\sqrt{3}+2\sqrt{5}+\sqrt{15}$
   (h) $1+\sqrt{3}-\sqrt{2}-\sqrt{6}$
   (i) $62$
   (j) $8+2\sqrt{15}$

Section 2

1. (a) $\frac{9\sqrt{2}+4}{20}$
   (b) $\frac{12\sqrt{3}+39}{35}$
   (c) $\frac{3\sqrt{2}}{2}$
   (d) $\frac{\sqrt{5}-28}{20\sqrt{2}}$
   (e) $\frac{7\sqrt{2}+23}{(\sqrt{2}+1)(\sqrt{2}+5)}$
   (f) $\frac{9\sqrt{3}+13\sqrt{7}}{(\sqrt{3}+\sqrt{7})(\sqrt{3}+2\sqrt{7})}$
   (g) $\frac{7\sqrt{5}+16}{6}$
   (h) $\frac{-7\sqrt{5}-25}{63}$
   (i) $\frac{-16+8\sqrt{2}+\sqrt{6}-\sqrt{3}}{(\sqrt{2}-1)(\sqrt{2}-3)}$
   (j) $\frac{6\sqrt{2}-42-5\sqrt{10}-5\sqrt{5}}{2\sqrt{5}}$

Section 3

1. (a) $\frac{3\sqrt{5}}{5}$
   (b) $\sqrt{2}$
   (c) $\frac{3\sqrt{3}}{4}$
   (d) $\frac{2+\sqrt{2}}{2}$
   (e) $\frac{\sqrt{5}(\sqrt{3}-1)}{5}$
   (f) $-\frac{2\sqrt{2}}{3}$
   (g) $\frac{\sqrt{10}(\sqrt{5}+3)}{10}$
   (h) $\frac{\sqrt{7}(\sqrt{2}-1)}{7}$
   (i) $\frac{\sqrt{3}+1}{2}$
   (j) $2(\sqrt{6}+2)$
   (k) $\frac{7(\sqrt{7}+2)}{3}$
   (l) $-\frac{3(\sqrt{5}-1)}{4}$
   (m) $\frac{\sqrt{5}(\sqrt{2}+3)}{5}$
   (n) $-\frac{(1+\sqrt{5})}{2}$
   (o) $-\frac{(\sqrt{3}+2)(\sqrt{3}-4)}{13}$
   (p) $\frac{(5+2\sqrt{3})(\sqrt{5}-3)}{2}$

**Exercises 2.5**

1. (a) $2$
   (b) $5$
   (c) $\sqrt{6}$
   (d) $8\sqrt{3}$
   (e) $6-2\sqrt{7}$
   (f) $-1-\sqrt{2}$
   (g) $\sqrt{3}-4$
   (h) $2\sqrt{2}$
   (i) $|a|$
   (j) $7b^2$
   (k) $\frac{p}{4}$
   (l) $\frac{25}{2}$

2.
(a) $13\sqrt{3}$
(b) $-12\sqrt{5}$
(c) $3\sqrt{5}$
(d) $20$
(e) $1$
(f) $30 + 12\sqrt{6}$
(g) $9 + 7\sqrt{5}$
(h) $5\sqrt{15} + 1$
(i) $\frac{2\sqrt{3}}{3}$
(j) $\frac{\sqrt{5}-1}{2}$
(k) $2 + \sqrt{3}$
(l) $\frac{10\sqrt{2}-3}{12}$
(m) $-3(\sqrt{3} + 3)$
(n) $-\frac{2}{7}(11\sqrt{2} - 9)$

3.
(a) $19 + 8\sqrt{3}$
(b) $34 + 9\sqrt{2}$
(c) 0.71
(d) 0.81
(e) 87.96 sq cm
(f) 1.83