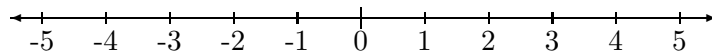


Worksheet 2.4 Introduction to Inequalities

Section 1 INEQUALITIES

The sign $<$ stands for less than. It was introduced so that we could write in shorthand things like 3 is less than 5. This becomes $3 < 5$. The sign $>$ stands for greater than. In a similar way we can write $5 > 3$ in this shorthand form. These statements are called inequalities. Recall the number line (introduced in an earlier worksheet):



It is drawn so that the numbers increase from left to right. Alternatively the numbers decrease from right to left. Any number which lies to the right of another on the number line is greater than it and any number which lies to the left of another on the number line is less than it.

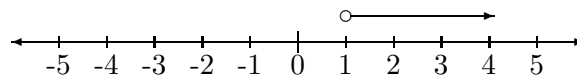
Example 1 :

- $-6 < 0$
- $5 > 2$
- $-5 < -4$
- $2.5 < 3$

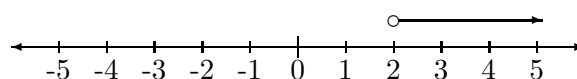
We can also use inequalities in algebraic expressions. So $21a$ is less than $30a$ is written

$$21a < 30a \quad (\text{only true if } a > 0)$$

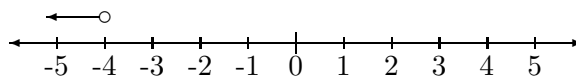
The expression $a > 1$ means that a is one of all the numbers to the right of 1 on the number line. We draw this using an open circle and an arrow heading to the right. The open circle sits over 1 and indicates that the actual number 1 is not included. That is a is bigger than 1 but not equal to 1:



Example 2 : $x > 2$



Example 3 : $x < -4$



This indicates all numbers to the left of -4 but not including -4 .

If we wish to include -4 in the above example we would write $x \leq -4$. This is shorthand for less than or equal to -4 .

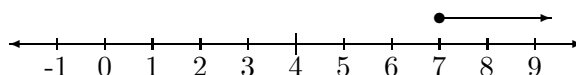
$$x \leq a \Rightarrow x < a \text{ or } x = a$$

and

$$x \geq a \Rightarrow x > a \text{ or } x = a$$

For \Rightarrow , read 'implies'. A closed circle or dot is used on the number line when the actual number is included in the inequality.

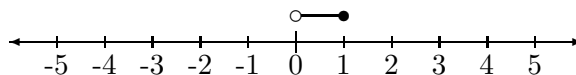
Example 4 : $x \geq 7$



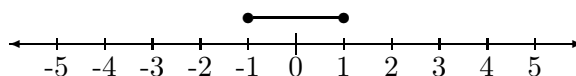
This indicates all the numbers to the right of 7 and 7 itself.

We can combine signs to make further algebraic expressions. For example $-3 \leq x < 2$ means all numbers greater than or equal to -3 but smaller than or equal to 2. That -3 and 2 and all the numbers in between. You may also see the notation $[-3, 2]$ used to represent this interval.

Example 5 : $0 < x \leq 1$ means all numbers greater than zero and less than or equal to one. In interval notation this is represented by $(0, 1]$. To draw this type of inequality on the number line we use a line drawn between the open or closed circles over the numbers. So $0 < x \leq 1$ is drawn as



Example 6 : $-1 \leq x \leq 1$



Example 7 : The number of matches in a box could be as little as 47 or as many as 58. If x stands for the number of matches in a box then we can write that

$$47 \leq x \leq 58$$

Exercises:

1. Write inequalities for the following:
 - (a) Numbers less than or equal to 6
 - (b) Numbers between -1 and 4 inclusive
 - (c) Numbers between 2 and 5 exclusive
2. Which of the numbers indicated satisfy the accompanying inequality?
 - (a) $x \geq 3$ $-1, 2, 3, 3\frac{1}{2}, 4\frac{1}{4}$
 - (b) $x < -4$ $-2, -7, -8\frac{1}{2}, 0$
 - (c) $-5 \leq x < 3$ $-6, -4, 0, \frac{1}{2}, 3$
 - (d) $2 < x \leq 5\frac{1}{2}$ $1\frac{1}{2}, 2\frac{1}{2}, 4, 5\frac{1}{4}, 5.5, 7$
3. Graph the following inequalities on number lines:
 - (a) $x \leq 1$
 - (b) $x > -4$
 - (c) $-1 < x < 5$

Section 2 SOLVING INEQUALITIES

Solving inequalities is similar to solving equations as in worksheet 2.2. We may add or subtract numbers or algebraic terms from both or all sides of the inequality to isolate the variable from the rest of the expression . Similarly we can multiply and divide each side with one very important qualification.

When multiplying or dividing both sides of an inequality by a negative number the sign must be reversed.

Example 1 : Given $-2x < 4$ we want to solve for x . Divide both sides by -2 and reverse the inequality. We get

$$\frac{-2x}{-2} > \frac{4}{-2}$$

and after cancelling we get

$$x > -2$$

Only do this when multiplying or dividing both sides of the inequality by a negative number. Let's see what happens when we don't do this.

Example 2 : Observe that $-1 < 2$ is a true statement.

$$\begin{aligned} -1 &< 2 \\ \frac{-1}{-1} &< \frac{2}{-1} \\ 1 &< -2 \end{aligned}$$

We have ended up with $1 < -2$, which is a false statement.

Some examples of solving inequalities follow.

Example 3 :

$$\begin{aligned} x + 3 &\geq 4 \\ x + 3 - 3 &\geq 4 - 3 \\ x &\geq 1 \end{aligned}$$

Example 4 :

$$\begin{aligned} x - 2 &\geq -5 \\ x - 2 + 2 &\geq -5 + 2 \\ x &\geq -3 \end{aligned}$$

Example 5 :

$$\begin{aligned} 5x &\geq 10 \\ \frac{5}{5}x &\geq \frac{10}{5} \\ x &\geq 2 \end{aligned}$$

Example 6 :

$$\begin{aligned}3 - 2x &\geq 5 \\-3 + 3 - 2x &\geq -3 + 5 \\-2x &\geq 2 \\\frac{-2x}{-2} &\leq \frac{2}{-2} \\x &\leq -1\end{aligned}$$

Example 7 :

$$\begin{aligned}5x + 3 &\leq 2x + 2 \\5x + 3 - 2x &\leq 2x + 2 - 2x \\3x + 3 &\leq 2 \\3x &\leq 2 - 3 \\3x &\leq -1 \\x &\leq \frac{-1}{3}\end{aligned}$$

Example 8 :

$$\begin{aligned}3\left(x + \frac{2}{3}\right) &< 5(2x + 5) \\3x + 2 &< 10x + 25 \\3x - 10x + 2 &< 25 \\-7x &< 25 - 2 \\-7x &< 23 \\x &> \frac{23}{-7} \\x &> -\frac{23}{7}\end{aligned}$$

Exercises:

1. Solve the following inequalities:

(a) $x - 7 < 5$

(d) $\frac{m+1}{2} \geq 4$

(b) $2x + 3 > 8$

(e) $6 - m < 8 - 4m$

(c) $4(m + 1) \leq m - 3$

(f) $3y + 2 \leq 5y + 10$

$$(g) \frac{y-1}{3} + 4 > \frac{y+1}{2}$$

$$(i) \frac{2x+1}{5} - \frac{x+3}{2} < 4x + 10$$

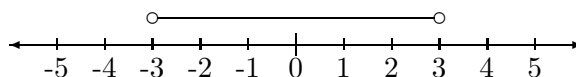
$$(h) 4(x + \frac{1}{2}) - 2(x + \frac{3}{2}) \leq 5$$

$$(j) 4(m + 3) + 5(2m - 1) > 7m + 6$$

Section 3 ABSOLUTE VALUES AND INEQUALITIES

The absolute value of a number was discussed in Worksheet 1.7. Recall that the absolute value of a , written $|a|$, is the distance in units that a is away from the origin. For example $|-7| = 7$. Alternatively you could define it as $|a| = +\sqrt{a^2}$. So when we combine absolute values and inequalities we are looking for all numbers that are either less than or greater than a certain distance away from the origin.

Example 1 : The equation $|x| < 3$ represents all the numbers whose distance away from the origin is less than 3 units. We could rewrite this inequality as $-3 < x < 3$. If we draw this on the number line we get

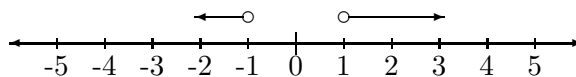


To solve inequalities involving absolute values we often need to rewrite the inequality without the absolute value signs as we have just done in the above example.

Example 2 : $|x| > 1$ can be rewritten as

$$x > 1 \quad \text{or} \quad x < -1$$

In other words all the numbers whose distance away from the origin is greater than 1 unit.



Notice that the solution is just the converse of $-1 \leq x \leq 1$. Since $|x| \leq 1$ is easier to solve than $|x| > 1$ we can begin by solving $-1 \leq x \leq 1$ and the solution will be its converse.

Example 3 :

$$\begin{aligned} |x| + 2 &\leq 4 \\ |x| + 2 - 2 &\leq 4 - 2 \\ |x| &\leq 2 \end{aligned}$$

this is the same as

$$-2 \leq x \leq 2$$

Example 4 :

$$\begin{aligned} |x + 3| &< 5 \\ -5 &< x + 3 < 5 \end{aligned}$$

Here we note that the expression inside the absolute value sign is treated as a single entity, as if it were in brackets, and must be rewritten before you try to solve it. So

$$\begin{aligned} -5 &< x + 3 &< 5 \\ -5 - 3 &< x + 3 - 3 &< 5 - 3 \\ -8 &< x &< 2 \end{aligned}$$

When dealing with a three-sided inequality like the above we follow the same rules as if it were two-sided, i.e. everything that we do to one side of the inequality we must do to all the other sides.

Example 5 : Solve:

$$|x - 2| \geq 4$$

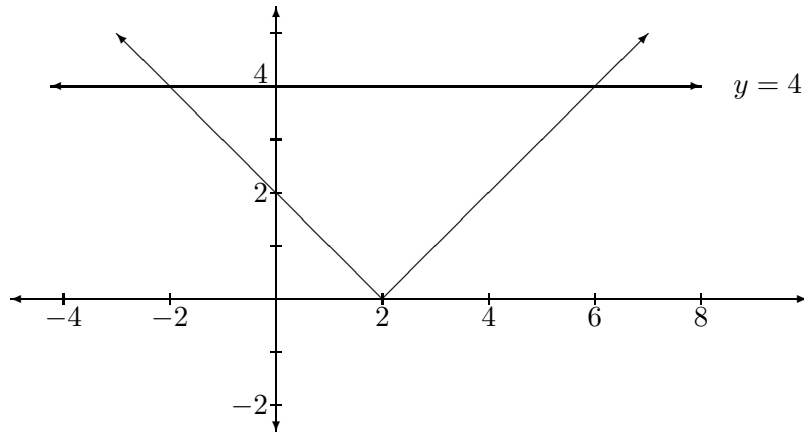
Consider the converse: $|x - 2| < 4$. Then

$$\begin{aligned} -4 &< x - 2 &< 4 \\ -2 &< x &< 6 \end{aligned}$$

Hence the solution is the converse

$$x \leq -2 \quad \text{or} \quad x \geq 6$$

This can also be viewed graphically:



Notice that the function $y = |x - 2|$ is at or above the line $y = 4$ for $x \leq -2$ and $x \geq 6$. To graph $y = |x - 2|$, we set up a table of values as follows:

x	0	1	4	-4
y	2	1	2	6

Exercises:

1. Solve the following inequalities, and graph the solutions on a number line:

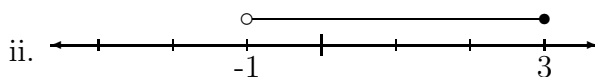
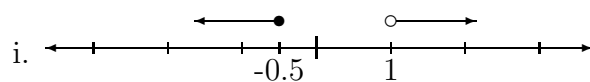
- (a) $|x| \leq 3$
- (b) $|x| > 4$
- (c) $|x| + 2 \leq 5$
- (d) $|x + 1| < 6$
- (e) $|x - 4| \geq 2$

Exercises 2.4 Introduction to Inequalities

1. (a) Show the following inequalities on a number line:

- i. $x > 3$
- ii. $x \leq -2.5$
- iii. $-1 < x \leq 4$
- iv. $x > 6$ or $x < -8$
- v. $[0, 5)$
- vi. $(-2, 4]$

(b) Write down the appropriate inequalities from the following information.



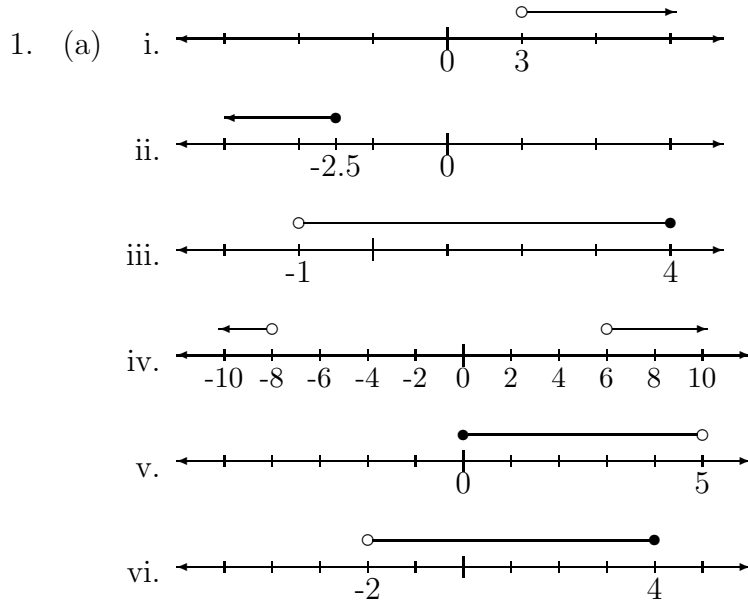
- iii. Sophie's blood alcohol level was at least 0.07
- iv. Tony had no more than 2 traffic offences.
- v. The number of drug offences was no more than 200.
- vi. To enter the theatre, you must be at least 18 years old or under 12 months old.
- vii. To join the veteran's team, you must be 35 or more.

- (c) i. Is $x = 1.5$ a solution to $7x + 5 > 19$?
 ii. Is $x = -2$ a solution to $4 - x < 6$?

2. Solve:

- | | |
|------------------------------|--------------------------------------------------|
| (a) $x - 3 \leq 2$ | (h) $\frac{1}{2}(x + 3) \geq \frac{3}{4}(x - 2)$ |
| (b) $-y > 6$ | (i) $ x = 7$ |
| (c) $2x < 24$ | (j) $ 6 - x = 7$ |
| (d) $-1 < x + 2 < 5$ | (k) $ y > 2$ |
| (e) $2a + 4 > 3a - 11$ | (l) $ 2x + 1 < 3$ |
| (f) $-2x + 7 > 10$ | (m) $ x - 5 \geq 5$ |
| (g) $3(t + 5) \leq 2(t + 1)$ | (n) $ x + 13 < -2$ |

Exercises 2.4



- (b) i. $x > 1$ or $x \leq -0.5$ v. $D \leq 200$
 ii. $-1 < x \leq 3$ vi. $x \geq 18$ or $x < 1$
 iii. $S \geq 0.07$
 iv. $T \leq 2$ vii. $A \geq 35$

- (c) (i) No (ii) No

2. (a) $x \leq 5$ (h) $x \leq 12$
 (b) $y < -6$ (i) $x = 7$ or $x = -7$
 (c) $x < 12$ (j) $x = -1$ or $x = 13$
 (d) $-3 < x < 3$ (k) $y > 2$ or $y < -2$
 (e) $a < 15$ (l) $-2 < x < 1$
 (f) $x < -\frac{3}{2}$ (m) $x \geq 10$ or $x \leq 0$
 (g) $t \leq -13$ (n) No solutions