Worksheet 2.1 Factors of Algebraic Expressions

Section 1 Finding Common Terms

In worksheet 1.2 we talked about factors of whole numbers. Remember, if $a \times b = ab$ then a is a factor of ab and b is a factor of ab. In a similar way we can look at the factors of an algebraic expression. So, for instance, $3uv$ has factors $1, 3, u, v$ and combinations of these like $3u, 3v, uv$ and of course $3uv$.

Other examples:-

1. $2ab = 2 \times a \times b$ has factors $1, 2, a, b, 2a, 2b, ab$ and $2ab$.
2. uvz has factors $u, v, z, uv, uz, vz, 1, uvz$.
3. xy has factors $1, x, y$ and xy.

In simplifying algebraic expressions we need to find common factors. Given two terms in an algebraic expression the common factors are those things which divide both of them.

Example 1 :

(a) $3uv$ and $6u$ have common factors $1, 3, u$ and $3u$.

(b) $2xy$ and $4xyz$ have common factors $1, 2, x, y, 2x, 2y, 2xy$ and xy.

The highest common factor is, as was the case with numbers, the biggest or largest factor that divides two expressions. So the highest common factor of $3uv$ and $6u$ (from example 1(a)) is $3u$; the highest common factor of $2xy$ and $4xyz$ (from example 1(b)) is $2xy$.

As with whole numbers we can also find the smallest algebraic expression that is a multiple of two expressions. This is called the lowest common multiple.

Example 2 :

$3uv$ and $6u$ have as their lowest common multiple $6uv$ since both $3uv$ and $6u$ divide into $6uv$ and do not both divide into a smaller expression.

To find the lowest common multiple we take the highest common factor and then multiply it by whatever is missing from each expression. From the example above $3uv$ and $6u$ have highest common factor $3u$. Now $3uv = 3u \times v$ and $6u = 3u \times 2$ so the lowest common multiple is $3u \times v \times 2 = 6uv$.

Example 3 : $9xy$ and $15xz$ have highest common factor $3x$.
$9xy = 3x \times 3y$ and $15xz = 3x \times 5z$ so the lowest common multiple is
$3x \times 3y \times 5z = 45xyz$.

Example 4 : $6a$ and $5b$ have highest common factor 1. So their lowest common multiple is $1 \times 6a \times 5b = 30ab$.

Exercises:

1. Find the highest common factor for each of the following:
   (a) $6x$, $18y$
   (b) $12mn$, $8m$
   (c) $3uv$, $4uw$
   (d) $18mp$, $9mn$
   (e) $27xyz$, $45xz$
2. Find the lowest common multiple for each of the following:
   (a) $6x$, $4xy$
   (b) $12xy$, $8xy$
   (c) $16mn$, $12np$
   (d) $24xyz$, $16yz$
   (e) $3m$, $45n$

Section 2 Simple Factoring

Sometimes in simplifying algebraic expressions or equations we would like to factorize them. This is a process which turns a sum into a product by removing common factors and placing them outside brackets. You have already seen an example of this.

When working out the perimeter of a paddock in the last worksheet we wanted two times the length plus two times the width. We wrote this as

$$P = 2l + 2w$$

By factorizing we can make a slightly tidier sum:

$$P = 2(l + w)$$

As 2 is a common factor to both terms it is placed outside of the brackets and the rest is left as a sum.

When factorizing algebraic expressions we look for the common factors in the terms and take these outside of the brackets to form a product as in the above example.

Example 1 :

$$9x + 24y = 3(3x + 8y)$$

Example 2 :

$$9x^2 + 3x + 15x^3 = 3(3x^2 + x + 5x^3)$$

But the terms inside the brackets still have x as a common factor:

$$9x^2 + 3x + 15x^3 = 3x(3x + 1 + 5x^2)$$

This is where we would stop since the terms inside the brackets have no further common factors.

Example 3 :

$$2ab^2 + ab^2c + 3ab = ab(2b + bc + 3)$$

Example 4 :

$$-2xy^2 - 4x^2y = -2xy(y + 2x)$$

Note that, by the laws mentioned in the last worksheet, the negative sign in front of the brackets will carry through the brackets, changing the sign of everything inside the brackets. Sometimes the common factor is not a simple multiple of numbers and letters but may in itself be a sum.

Example 5 : Simplify $5(x + 2) + y(x + 2) = (5 + y)(x + 2)$. We note that $(x + 2)$ is a common factor, so we put $(x + 2)$ out the front:

$$5(x + 2) + y(x + 2) = (x + 2)(5 + y)$$

Example 6 : $7(y + 1) - x(y + 1) = (y + 1)(7 - x)$

Exercises:

1. Factorize the following expressions:

   (a) $7x + 4$
   (b) $20x - 10$
   (c) $18xy - 3yz$
   (d) $12mn + 18mp$
   (e) $16m^2 - 4m$
   (f) $3x^2 + 6x - 18$
   (g) $-6x - 24$
   (h) $-2xy - 8x$
   (i) $24mn - 16m^2n$
   (j) $-x^2y - y^2x$
   (k) $12m^2n + 24m^2n^2$
   (l) $72y^2p - 18y^3p^2$

2. Factorize the following expressions:

   (a) $4(x + 3) + m(x + 3)$
   (b) $x(x - 1) + 5(x - 1)$
   (c) $y(y + 4) - 6(y + 4)$
   (d) $x^2(x + 7) + x(x + 7)$
   (e) $3x(x - 4) - 7(x - 4)$

Section 3 Algebraic Fractions

One use of the factorization of algebraic expressions or of being able to find common algebraic factors is to simplify algebraic fractions. Using the same method as with ordinary fractions we can cancel out common factors in algebraic fractions to make a simpler equivalent fraction.

Example 1 :

$$\frac{x}{2x} = \frac{1\times \not{x}}{2\times \not{x}} = \frac{1}{2}$$

by cancelling out the common x in the numerator and denominator.

Example 2 :

$$\frac{5x^2y}{15xy} = \frac{5xy \times x}{5xy \times 3} = \frac{x}{3}$$

noting that $5xy$ is a common factor.

Example 3 :

$$\frac{4a + 2ab}{2a} = \frac{2a(2+b)}{2a} = 2 + b$$

noting that $2a$ is a factor in common for the two terms in the sum and then cancelling.

Example 4 :

$$\begin{aligned}
\frac{7x^2}{5y} \times \frac{15yz}{x} &= \frac{x \times 7x}{5y} \times \frac{5y \times 3z}{x} \\
&= \frac{\cancel{x} \times 7x \times \cancel{5}y \times 3z}{\cancel{5}y \times \cancel{x}} \\
&= 7x \times 3z \\
&= 21xz
\end{aligned}$$

Note that, when either the numerator or denominator are completely cancelled, they become 1, not 0.

Example 5 :

$$\begin{aligned}
\frac{x}{3} \div \frac{2x^2}{3} &= \frac{x}{3} \times \frac{3}{2x^2} \\
&= \frac{1}{2x}
\end{aligned}$$

Example 6 :

$$\begin{aligned}
\frac{6x+18}{20} \div \frac{3x+9}{15} &= \frac{6(x+3)}{20} \times \frac{15}{3(x+3)} \\
&= \frac{6 \times 15}{20 \times 3} \\
&= \frac{3 \times \cancel{2} \times \cancel{5} \times \cancel{3}}{\cancel{5} \times 2 \times \cancel{2} \times \cancel{3}} \\
&= \frac{3}{2} \\
&= 1\frac{1}{2}
\end{aligned}$$

Exercises:

1. Simplify the following:
   (a) $\frac{3x}{15}$
   (b) $\frac{2x+10}{4}$
   (c) $\frac{5x+20}{x+4}$
   (d) $\frac{x^2-4x}{x-4}$
   (e) $\frac{3x^2-9x}{2x-6}$
   (f) $\frac{9x+27}{9x+18}$
   (g) $\frac{6ab+2a}{2b}$
   (h) $\frac{16m^2n-8mn}{12m-6}$
   (i) $\frac{4mnp-8mp}{12mn}$

2. Simplify the following:
   (a) $\frac{3x+9}{14} \times \frac{7x+21}{x+3}$
   (b) $\frac{x^2-5x}{2x+10} \times \frac{3x+15}{4x}$
   (c) $\frac{3mp+4p}{8p} \times \frac{12p^2}{3m+4}$
   (d) $\frac{16}{2mp+4m} \times \frac{6m^2+8m}{12}$
   (e) $\frac{24x-8}{12} \div \frac{9x-3}{6}$
   (f) $\frac{x^2+2x}{5} \div \frac{2x+4}{20}$
   (g) $\frac{p^2+pq}{7p} \div \frac{8p+8q}{21q}$
   (h) $\frac{5xy-15y}{4x-12} \div \frac{6y^2}{x+y}$

Exercises 2.1 Factors of Algebraic Expressions

1. What are the factors of 18?

2. What are the common factors of:

   (a) 16 and 24

   (b) $6tm$ and $14t^2$

3. What is the highest common factor of

   (a) 12 and 32

   (b) 24 and 40

   (c) 5 and 13

4. Factorize the following algebraic expressions

   (a) $-3x + 21$

   (b) $6x^2 + 3x$

   (c) $18x^2 + 12xy$

   (d) $6tm - 24m^2$

   (e) $8x + 12y + 10x + 15y$

   (f) $x^2 - 7x + 3x - 21$

5. Simplify the following

   (a) $\frac{3x+12}{3}$

   (b) $\frac{6xy+18x}{12}$

   (c) $\frac{7mn}{24} \times \frac{8}{m}$

   (d) $\frac{4x+20}{5} \div \frac{8x+40}{20}$

   (e) $\frac{6xy^2}{7} \times \frac{21x^2}{y} \div \frac{32xy^2}{91}$

   (f) $\frac{12p^2q^2}{5} \times \frac{15}{4pq} \div 3$

Answers 2.1

Section 1

1. (a) 6 (b) $4m$ (c) u (d) $9m$ (e) $9xz$
2. (a) $12xy$ (b) $24xy$ (c) $48mnp$ (d) $48xyz$ (e) $45mn$

Section 2

1. (a) $7x + 4$ (e) $4m(4m - 1)$ (i) $8mn(3 - 2m)$
   (b) $10(2x - 1)$ (f) $3(x^2 + 2x - 9)$ (j) $-xy(x + y)$
   (c) $3y(6x - z)$ (g) $-6(x + 4)$ (k) $12m^2n(1 + 2n)$
   (d) $6m(2n + 3p)$ (h) $-2x(y + 4)$ (l) $18y^2p(4 - p)$
2. (a) $(x + 3)(4 + m)$ (c) $(y + 4)(y - 6)$ (e) $(x - 4)(3x - 7)$
   (b) $(x - 1)(x + 5)$ (d) $x(x + 7)(x + 1)$

Section 3

1. (a) $\frac{x}{5}$ (d) x (g) $\frac{a(3b+1)}{b}$
   (b) $\frac{x+5}{2}$ (e) $\frac{3x}{2}$ (h) $\frac{4mn}{3}$
   (c) 5 (f) $\frac{x+3}{x+2}$ (i) $\frac{p(n-2)}{3n}$
2. (a) $\frac{3(x+3)}{2}$ (c) $\frac{3p^2}{2}$ (e) $\frac{4}{3}$ (g) $\frac{3q}{8}$
   (b) $\frac{3(x-5)}{8}$ (d) $\frac{4(3m+4)}{3(p+2)}$ (f) $2x$ (h) $\frac{5(x+y)}{24y}$

Exercises 2.1

1. 1, 2, 3, 6, 9, 18
2. (a) 1, 2, 4, 8
   (b) 1, 2, t, $2t$
3. (a) 4 (b) 8 (c) 1

4. (a) $-3(x-7)$
   (b) $3x(2x+1)$
   (c) $6x(3x+2y)$
   (d) $6m(t-4m)$
   (e) $9(2x+3y)$
   (f) $(x+3)(x-7)$

5. (a) $x+4$
   (b) $\frac{x(y+3)}{2}$
   (c) $\frac{7n}{3}$
   (d) 2
   (e) $\frac{819x^2}{16y}$
   (f) $3pq$