Table of Contents

Test Four

This is a self-diagnostic test. Each question relates to a worksheet in a series available in the MUMS the WORD series. For example question 4 relates to worksheet 4.4 *Applications of Integration*. If you score 100% on this test and test 3 then we feel you are adequately prepared for your first year mathematics course. For those of you who had trouble with a few of the questions, we recommend working through the appproriate worksheets and associated computer aided learning packages in this series.

---

1. (a) Differentiate $y = \log(3x+2)$

   (b) Find $\frac{dy}{dx}$ if $y = x^2 \cos x$

2. (a) Given the following monotonically increasing function, find an upper and lower limit for the area under the curve between 0 and 4.

x	0	1	2	3	4
$g(x)$	2	3	5	6.5	7

   (b) Find the area under the curve $y = x^2 + 1$ between $x = 1$ and $x = 3$.

3. Evaluate the following indefinite integrals:

   (a) $\displaystyle\int \frac{1}{x}\, dx$

   (b) $\displaystyle\int \sec^2 x\, dx$

4. (a) Given $\dfrac{d^2x}{dt^2} = 9$ for all x and when $t = 0$ we have $\frac{dx}{dt} = 4$ and $x = 3$. What is x as a function of t?

   (b) A population $P(t)$ is given by the following formula:

$$P(t) = P(0)e^{kt}$$

   If the initial population is 1000, and the growth rate is 0.01, what is the population at $t = 100$? (You can leave the answer in terms of the natural exponential)

5. Divide $6x^3 + x^2 - x + 4$ by $x + 1$.

6. (a) Simplify $\dfrac{\sin 4x}{(\cos^2 x - \sin^2 x)\sin x \cos x}$.

(b) Find the exact value of $\cos\left(\frac{\pi}{8}\right)$.

7. Sketch $y = 2\sqrt{x-3} + 1$.

8. Let $f(x) = \frac{x+1}{x+2}$ and $g(x) = \sqrt{x}$.

   (a) Find $(f \circ g)(x)$.

   (b) Find $f^{-1}(x)$.

9. Let $f(x) = \frac{1}{e^x - 3}$.

   (a) Find the largest domain of f.

   (b) Find the inverse of f.

10. (a) Evaluate $\frac{6!}{4!2!}$.

    (b) Write out the sum $\sum_{n=1}^{5} n^3$ without using sigma notation.

    (c) Write the sum $x^2 + 2x^4 + 3x^6 + ... + 10x^{20}$ in sigma notation.

11. How many 3-digit numbers can be formed from the digits $1, 2, 3, 4, 5, 6$ if repetition of digits are (i) allowed, (ii) not allowed.

12. What is the coefficient of x^2 in the expansion of $(5x - 1)^5$?

13. Use Mathematical Induction to prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

for all $n \in \mathbb{N}$.

Worksheet 4.1 More Differentiation

Section 1 The chain rule

In the last worksheet, you were shown how to find the derivative of functions like $e^{f(x)}$ and $\sin g(x)$. This section gives a method of differentiating those functions which are what we call composite functions. The method is called the chain rule. The chain rule allows us to differentiate composite functions. Composite functions are functions of functions, and can be written as

$$g(x) = f(u(x))$$

So if $u(x) = x^2$ and $f(u) = \cos u$, then

$$f(u(x)) = \cos x^2$$

The derivative of such functions is given by the following rule:

$$g'(x) = \frac{du(x)}{dx} \times \frac{df}{du}$$

So for our example of $g(x) = f(u(x)) = \cos x^2$ we have

$$\frac{df}{du} = -\sin u = -\sin x^2 \quad \text{and} \quad \frac{dg}{dx} = (2x) \times (-\sin x^2)$$

The trick is working out which function is the f and which is the u – it is what you do to the input first.

Example 1 : Differentiate e^{5x^2}. Let $u(x) = 5x^2$ and $f(u) = e^u$. If $g(x) = f(u(x))$ then

$$g'(x) = u'(x) \times \frac{df}{du}$$

We have $\frac{du}{dx} = 10x$ and $\frac{df}{du} = e^u = e^{5x^2}$ so that

$$g'(x) = 10xe^{5x^2}$$

Example 2 : Differentiate $g(x) = \sin(e^x)$. We let $u(x) = e^x$ and $f(u) = \sin u$. Then

$$\begin{aligned} u'(x) &= e^x \\ \frac{df}{du} &= \cos u = \cos e^x \\ \frac{dg}{dx} &= u'(x) \times \frac{df}{du} \\ &= e^x \cos(e^x) \end{aligned}$$

Example 3 : Differentiate $y = (6x^2 + 3)^4$. We let $u(x) = 6x^2 + 3$ and $f(u) = u^4$. Then

$$\begin{aligned} u'(x) &= 12x \\ \frac{df}{du} &= 4u^3 = 4(6x^2+3)^3 \\ \frac{dy}{dx} &= u'(x) \times f'(u) \\ &= 12x \times 4(6x^2+3)^3 \end{aligned}$$

Example 4 : Differentiate $y = (3x + 2)^4$. Let $u(x) = 3x + 2$. Then

$$\frac{dy}{dx} = 3 \times 4 \times (3x+2)^3 = 12(3x+2)^3.$$

Exercises:

1. Differentiate the following with respect to x.

   (a) $\sin 3x$

   (b) $\tan(-2x)$

   (c) $\cos 6x^2$

   (d) $(4x + 5)^5$

   (e) $(6x - 1)^3$

   (f) $(3x^2 + 1)^4$

   (g) e^{4x}

   (h) $7e^{2x}$

   (i) $e^{\sin x}$

   (j) $e^{\cos x}$

   (k) $(6 - 2x)^3$

   (l) $(7 - x)^4$

Section 2 The product rule

The product rule gives us a method of working out the derivative of a function which can be written as the product of functions. Examples of such functions are $x^2 \sin x$, $5x \log x$, and $e^x \cos x$. These functions all have the general form

$$\begin{aligned} h(x) &= f(x)g(x) \quad \text{or in simpler terms} \\ h &= fg \end{aligned}$$

For functions that are written in this form, the product rule says:

$$\frac{dh}{dx} = \frac{df(x)}{dx} g(x) + f(x) \frac{dg(x)}{dx} \quad \text{or} \quad h' = f'g + fg'$$

When first working with the product rule, it is wise to write down all the steps in the calculation to avoid any confusion.

Example 1 : Differentiate $h(x) = x^2 \sin x$.
Let $f(x) = x^2$ and $g(x) = \sin x$. Then $f'(x) = 2x$ and $g'(x) = \cos x$, which gives

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 2x \sin x + x^2 \cos x \end{aligned}$$

Note that in terms such as $\cos x \times x^2$, it is less ambiguous to write $x^2 \cos x$ to make it clear that we are not taking the cos of the x^2 term.

Example 2 : Differentiate $h(x) = 5x \log x$.
Let $f(x) = 5x$ and $g(x) = \log x$ so that $f'(x) = 5$ and $g'(x) = \frac{1}{x}$. Then

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 5 \log x + 5x \times \frac{1}{x} \\ &= 5 \log x + 5 \end{aligned}$$

Example 3 : Differentiate $p(x) = e^x \cos x$.
Let $a(x) = e^x$ and $b(x) = \cos x$. Then $a'(x) = e^x$ and $b'(x) = -\sin x$, so that

$$\begin{aligned} p'(x) &= a'(x)b(x) + a(x)b'(x) \\ &= e^x \cos x + e^x(-\sin x) \\ &= e^x(\cos x - \sin x) \end{aligned}$$

Example 4 : Differentiate $h(x) = 3x^2 e^x$.
Let $f(x) = 3x^2$ and $g(x) = e^x$ so that $f'(x) = 6x$ and $g'(x) = e^x$. Then

$$\begin{aligned} h'(x) &= 6xe^x + 3x^2 e^x \\ &= 3x(2e^x + xe^x) \\ &= 3x(x+2)e^x \end{aligned}$$

Note that, when using the product rule, it makes no difference which part of the whole function we call $f(x)$ or $g(x)$ (so long as we are able to differentiate the f or g that we choose). So in example 4, we could have let $f(x) = 6e^x$ and $g(x) = x$ and the final result for $h'(x)$ would have been the same.

Exercises:

1. Differentiate the following with respect to x.

   (a) $x^2 \sin x$

   (b) $4xe^{3x}$

   (c) x^2e^{3x}

   (d) $x \cos x$

   (e) $4x \log(2x + 1)$

   (f) $x^2 \log(x + 2)$

   (g) $x^2e^{x^3}$

   (h) $(3x + 1)(x + 1)^3$

   (i) $3x(x + 2)^3$

   (j) $(4x - 1)e^{2x}$

   (k) $\sin xe^{2x}$

   (l) $\cos(2x)e^{4x}$

Section 3 The quotient rule

The quotient rule is the last rule for differentiation that will be discussed in these worksheets. The quotient rule is derived from the product rule and the chain rule; the derivation is given at the end of the worksheet for those that are interested. The quotient rule helps to differentiate functions like $\frac{e^{2x}}{x^2}$, $\frac{x^2}{\cos x}$ and $\frac{x^2+1}{x^3+3}$. The general form of such expressions is given by $k(x) = \frac{u(x)}{v(x)}$, and the quotient rule says that

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \quad \text{or}$$

$$k' = \frac{u'v - uv'}{v^2}$$

It is a good idea to do some bookkeeping when using the quotient rule.

Example 1 : Differentiate $k(x) = \frac{e^{2x}}{x^2}$.
Let $u(x) = e^{2x}$ and $v(x) = x^2$. Then $u'(x) = 2e^{2x}$ and $v'(x) = 2x$, which gives

$$\begin{aligned} k'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \\ &= \frac{2e^{2x}x^2 - e^{2x}2x}{(x^2)^2} \\ &= \frac{2xe^{2x}(x - 1)}{x^4} \\ &= \frac{2e^{2x}(x - 1)}{x^3} \end{aligned}$$

Note that the choice of $u(x)$ and $v(x)$ are not interchangeable as in the product rule. Given the complicated appearance of the quotient rule, it is wise to be consistent and always let $u(x)$ be the numerator and $v(x)$ the denominator.

Example 2 : Differentiate $p(x) = \frac{x^2}{\cos x}$. Let $u(x) = x^2$ and $v(x) = \cos x$. Then $u'(x) = 2x$ and $v'(x) = -\sin x$, so that

$$\begin{aligned} p'(x) &= \frac{2x\cos x - x^2(-\sin x)}{(\cos x)^2} \\ &= \frac{2x\cos x + x^2\sin x}{\cos^2 x} \\ &= \frac{x(2\cos x + x\sin x)}{\cos^2 x} \end{aligned}$$

Example 3 : Differentiate $p(x) = \frac{x^2+1}{x^3+3}$.
Let $u(x) = x^2 + 1$ and $v(x) = x^3 + 3$. Then $u'(x) = 2x$ and $v'(x) = 3x^2$, so that

$$\begin{aligned} p'(x) &= \frac{2x(x^3+3) - (x^2+1)3x^2}{(x^3+3)^2} \\ &= \frac{6x - 3x^2 - x^4}{(x^3+3)^2} \end{aligned}$$

We now derive the quotient rule from the product and chain rule; skip the derivation if you don't feel the need to know. Let

$$k(x) = \frac{u(x)}{v(x)} = u(x)(v(x))^{-1}$$

We now use the product rule and let $f(x) = u(x)$ and $g(x) = (v(x))^{-1}$. Then $f'(x) = u'(x)$ and the derivative of $g(x)$ is given by the chain rule:

$$\begin{aligned} g'(x) &= -1(v'(x))(v(x))^{-2} \\ &= \frac{-v'(x)}{(v(x))^2} \end{aligned}$$

Using the product rule on $k(x)$ (the thing we are trying to differentiate), we get

$$\begin{aligned} k'(x) &= u'(x)(v(x))^{-1} + u(x) \times \frac{-v'(x)}{(v(x))^2} \\ &= \frac{u'(x)}{v(x)} - \frac{u(x)v'(x)}{(v(x))^2} \\ &= \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \end{aligned}$$

This is the quotient rule.

Exercises:

1. Differentiate the following with respect to x.

(a) $\dfrac{x^3}{x^2+1}$

(b) $\dfrac{x^2+3}{x+1}$

(c) $\dfrac{x-1}{2x+3}$

(d) $\dfrac{e^{2x}}{x-3}$

(e) $\dfrac{\sin x}{x^2}$

(f) $\dfrac{\sin x}{\cos x}$

(g) $\dfrac{3x}{x^2-2}$

(h) $\dfrac{x+6}{x-4}$

(i) $\dfrac{6e^x}{x+5}$

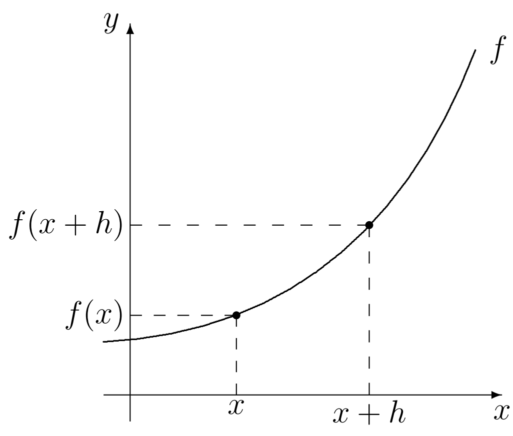
(j) $\dfrac{e^{2x}}{\sin x}$

Section 4 Equations of tangents and normals to curves

When the topic of differentiation was first introduced in section 1 of Worksheet 3.8, we said that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

This was motivated from this picture:

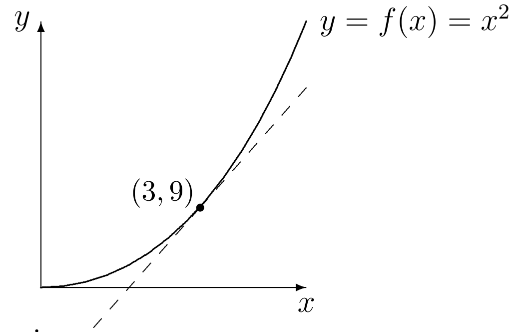


As $h \to 0$ the secant joining $(x, f(x))$ and $(x+h, f(x+h))$ becomes a better and better approximation to the tangent of f at the point $(x, f(x))$. We will now find the equation of this tangent. Recall that the tangent is just a straight line and it passes through the point $(x, f(x))$ on the curve. We have already found the equation of a straight line through a given point, say (x_1, y_1), with a given slope, say m – this was done in Worksheet 2.10. The equation of such a straight line is

$$y - y_1 = m(x - x_1).$$

Example 1 : Find the equation of the tangent to the curve $y = x^2$ at the point $(3, 9)$.

A piece of the function is drawn as well as the tangent.



The derivative of the function is

$$\frac{dy}{dx} = 2x.$$

At the point $(3, 9)$, $\frac{dy}{dx} = 2 \times 3 = 6$ so that the slope of the tangent line is 6. Now, a point that lies on the tangent line is $(3, 9)$, so the equation of the tangent line is

$$\begin{aligned} y - 9 &= 6(x - 3) \\ y - 9 &= 6x - 18 \\ y &= 6x - 9 \end{aligned}$$

The equation of the tangent of $y = x^2$ at $(3, 9)$ is $y = 6x - 9$.

Example 2 : Find the equation of the tangent to the curve $y = x^3 - x + 4$ at the point $(1, 4)$.

We will find the equation without drawing the graph. We have

$$\frac{dy}{dx} = 3x^2 - 1,$$

so the slope of the tangent at $x = 1$ is $3(1)^2 - 1 = 2$. A point that the tangent passes through is $(1, 4)$, so the equation must be given by

$$\begin{aligned} y - 4 &= 2(x - 1) \\ y - 4 &= 2x - 2 \\ y &= 2x + 2 \end{aligned}$$

The equation of the tangent of $y = x^3 - x + 4$ at $(1, 4)$ is $y = 2x + 2$.

Example 3 : Find the equation of the tangent to the curve $y = e^{2x}$ at the point $(3, e^6)$.

We have

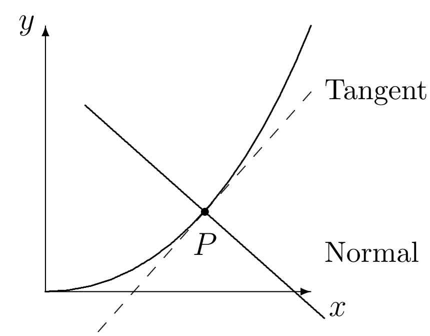
$$\frac{dy}{dx} = 2e^{2x},$$

so the slope of the tangent at $x = 3$ is $2e^6$. A point that the tangent passes through is $(3, e^6)$, so the equation must be given by

$$\begin{aligned} y - e^6 &= 2e^6(x-3) \\ y - e^6 &= 2e^6 x - 6e^6 \\ y &= 2e^6 x - 5e^6 \\ y &= e^6(2x-5) \end{aligned}$$

The equation of the tangent of $y = e^{2x}$ at $(3, e^6)$ is $y = e^6(2x-5)$.

The *normal* to a curve at a particular point is the straight line that passes through the point in question on the curve and is perpendicular to the tangent to the curve.



Example 4 : Find the equation of the normal to the curve $y = x^2$ at the point $(3, 9)$.

From example 1, the slope of the tangent is 6, so the gradient of the normal to the tangent is $-\frac{1}{6}$. (Recall that in section 3 of Worksheet 2.10 we said that if two lines are perpendicular, then the product of their slopes is -1.) So the equation of the normal at the point $(3, 9)$ is

$$\begin{aligned} y - 9 &= -\frac{1}{6}(x-3) \\ y - 9 &= -\frac{1}{6}x + \frac{1}{2} \\ y &= -\frac{1}{6}x + \frac{19}{2} \end{aligned}$$

Example 5 : Find the equations of the tangent and normal to the curve $y = x^3 - 5x + 6$ at $(-3, -6)$.

We have $\dfrac{dy}{dx} = 3x^2 - 5$, so the slope of the tangent when $x = -3$ is 22. The equation of the tangent is then given by

$$\begin{aligned} y - (-6) &= 22(x - (-3)) \\ y &= 22x + 60 \end{aligned}$$

The equation of the normal is given by

$$\begin{aligned} y - (-6) &= -\frac{1}{22}(x - (-3)) \\ y &= -\frac{1}{22}x - \frac{135}{22} \end{aligned}$$

Exercises:

1. Find the equation of the tangent to the curve
   (a) $y = x^2 - 4x + 6$ at the point $(-2, 18)$
   (b) $y = 6 - x^2$ when $x = 3$
   (c) $y = x^3 - 4x + 30$ when $x = -5$
2. Find the equation normal to the curve
   (a) $y = 8 - 3x^2$ at the point $(4, -40)$
   (b) $y = x^3 - 2x^2 + 6$ when $x = -1$
   (c) $y = 6/x$ at the point $(-2, -3)$.
3. Find the equation of the tangent to the curve $y = 3x^2 - 2x + 4$ at the point $(1, 5)$ and also find the point where the tangent cuts the x axis.

Exercises for Worksheet 4.1

1. Differentiate the following

   (a) $y = \frac{1}{x^2} - 6x + 4$

   (b) $y = xe^{2x}$

   (c) $y = \sin 2x - \cos 4x$

   (d) $y = (2x+1)^3(x+2)$

   (e) $y = 4x \sin x$

   (f) $y = \log(x^2+1)$

   (g) $y = x \log x$

   (h) $y = (\sin x)^2$

   (i) $y = e^x \sin x$

   (j) $y = \frac{6x+1}{x-4}$

   (k) $y = \frac{x^2}{x+3}$

   (l) $y = \frac{e^x}{x-2}$

2. (a) Find the equation of the tangent to the curve $y = x^2 \log x$ at the point (e, e^2).

   (b) If $f(x) = \sin 2x - \cos 4x$, find $f'(\frac{\pi}{4})$.

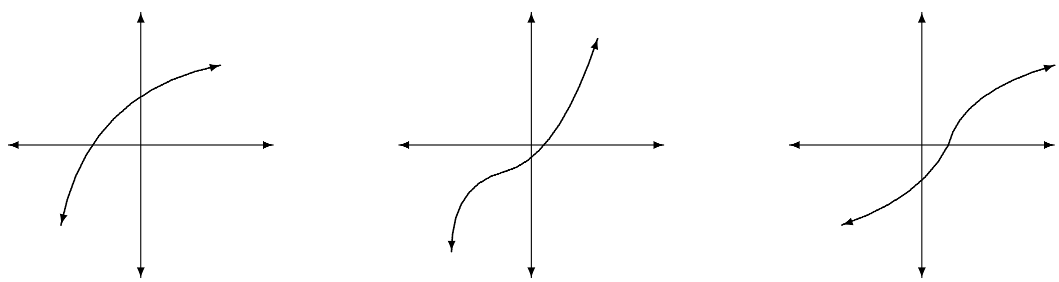
   (c) If $y = (x^2-1)(1+x)$, show that $x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2x - 2 = 0$.

   (d) Find the turning point of the curve $y = x^2 + 3x - 4$ and state whether it is a maximum or minimum turning point.

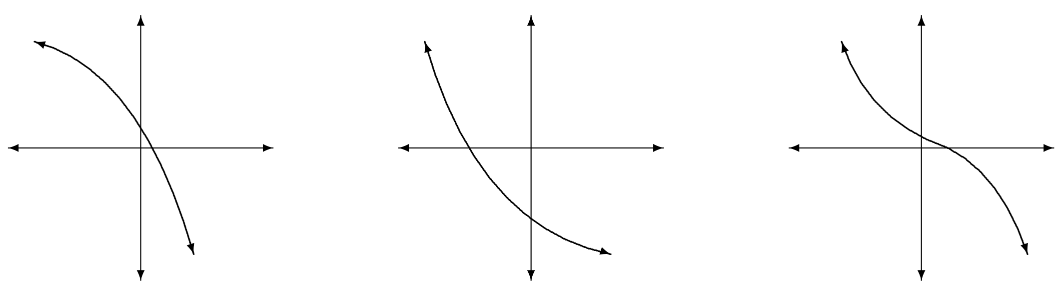
Worksheet 4.2 Introduction to Integration

Section 1 Left and Right Rectangles

A function f which has the property that if $b > a$ then $f(b) > f(a)$ is called monotonically increasing - as the input increases, then the output increases. The 'monotonically' part comes from the property that there are no maxima or minima. The slope of a monotonically increasing function will always be greater than or equal to zero, and it will only equal zero at a point of inflection. Here are some examples of the graphs of monotonically increasing functions:

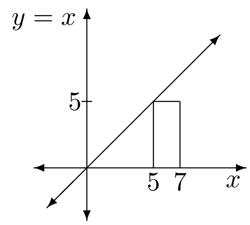


A monotonically decreasing function f is one which has the property that if $b > a$, then $f(b) < f(a)$. In other words, as the input gets larger, the output gets smaller. The slope of a monotonically decreasing function is always less than or equal to zero, and is only zero at a point of inflection. Here are some examples of the graphs of monotonically decreasing functions:



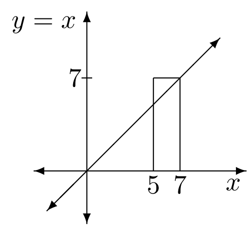
Sometimes it is important for us to be able to estimate the area under a curve, which might represent a quantity in which we are interested. For example, if we had a graph of a motorist's velocity as a function of time for a journey that lasted an hour, the area under the curve would represent the distance travelled over the journey. We now give a method that can be used to estimate the area under certain types of curves, namely those that are either monotonically increasing or decreasing. Note that all functions can be broken up into a sequence of parts, each of which is either (a) monotonically increasing or (b) monotonically decreasing or (c) horizontal. A vertical line is not a function as it does not have the property that each input value has one and only one output value. The estimation method involves splitting the area up into rectangles to give a lower and upper bound to the area under the curve.

Take the function $y = x$ as an example, and say we wish to know the area under the curve $y = x$ between $x = 5$ and $x = 7$. First we draw the graph:



If we draw a rectangle the height of the value of the function at $x = 5$ which stretches across to the same height above $x = 7$ we get the shaded region. This is called a left rectangle, as its height is given by the function value on the left hand side of the interval. Since $f(5) = 5$, the height of the rectangle is 5, and the width is $7 - 5 = 2$, so the area of the rectangle is $5 \times 2 = 10$.

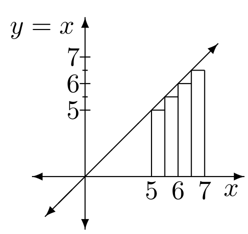
Now we draw the following diagram:

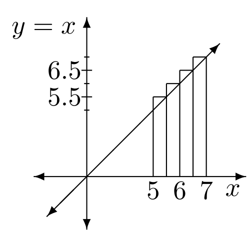


For the height of the rectangle, we use the value of the function at $x = 7$, which is $f(7) = 7$. This is called a right rectangle, and it has area $7 \times 2 = 14$. The area under the line $y = x$ between $x = 5$ and $x = 7$ must lie somewhere between 10 and 14 since the area of the right rectangle is bigger than the area under the line, and the area of the left rectangle is smaller than than the area under the line. Thus, if A is the area then,

$$10 < A < 14$$

We can get a closer approximation to the area under this line by breaking the interval into smaller pieces. Say we look at the function $y = x$ at every 1/2 unit, and add up the area of the rectangles formed by using as our intervals: [5,5.5],[5.5,6],[6,6.5], and [6.5,7]. The left rectangles are shown in figure 3, and the right rectangles in figure 4.





Note : The area of each rectangle is found by multiplying the base by the height. The area given by the left rectangles is the sum:

$$f(5)(5.5-5)+f(5.5)(6-5.5)+f(6)(6.5-6)+f(6.5)(7-6.5)=\frac{5}{2}+\frac{11}{4}+\frac{6}{2}+\frac{13}{4}=11\frac{1}{2}$$

The area given by the right rectangles is

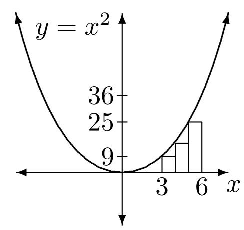
$$f(5.5)\frac{1}{2}+f(6)\frac{1}{2}+f(6.5)\frac{1}{2}+f(7)\frac{1}{2}=\frac{11}{4}+\frac{6}{2}+\frac{13}{4}+\frac{7}{2}=12\frac{1}{2}$$

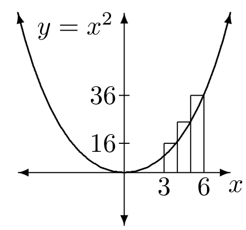
The lower and upper bounds on the area A are now given by:

$$11\frac{1}{2}<A<12\frac{1}{2}$$

By taking smaller and smaller intervals, we are going to bring the lower and upper bounds closer and closer together and so get a better approximation to the actual area.

Example 1 : Estimate the area under the curve $y=x^2$ from $x=3$ to $x=6$ by splitting the interval into 3 parts. Note that $y=x^2$ is monotonically increasing in the interval that we are interested in.





The left rectangles give a lower bound on the area:

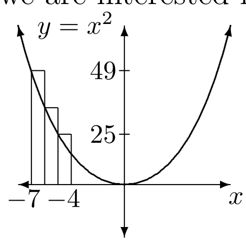
$$A_L=f(3)\times 1+f(4)\times 1+f(5)\times 1=9+16+25=50$$

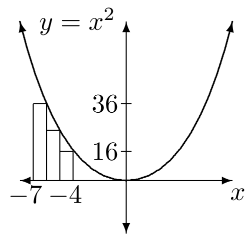
The right rectangles give an upper bound on the area:

$$A_R=f(4)\times 1+f(5)\times 1+f(6)\times 1=16+25+36=77$$

Therefore $50<A<77$. This is not a good approximation, but by taking smaller and smaller intervals, the error will be reduced.

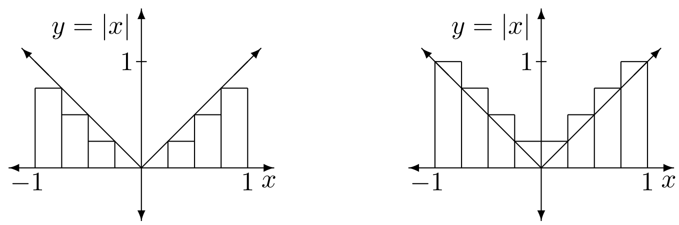
Example 2 : Estimate the area under the curve $y=x^2$ from $x=-7$ to $x=-4$ by splitting the interval into 3 parts. Note that $y=x^2$ is monotonically decreasing in the interval that we are interested in.





The left rectangles give $A_L = f(-7)\times 1 + f(-6)\times 1 + f(-5)\times 1 = 49+36+25 = 110$. The right rectangles give $A_R = f(-6)\times 1 + f(-5)\times 1 + f(-4)\times 1 = 36+25+16 = 77$. Notice that A_R is smaller than A_L. This is because now we are looking at a monotonically decreasing function, so A_R sets a lower bound and A_L an upper bound.

Example 3 : Find the area under the curve $y = |x|$ in the interval -1 to 1. Use 8 subintervals.



Since the interval is not monotonically increasing or decreasing, we need to split it up into pieces that are. So first we look at from -1 to 0, which yields the following:

$$\begin{aligned} A_L(-1,0) &= f(-1)\frac{1}{4} + f(-\frac{3}{4})\frac{1}{4} + f(-\frac{1}{2})\frac{1}{4} + f(-\frac{1}{4})\frac{1}{4} = \frac{4+3+2+1}{16} = \frac{5}{8} \\ A_R(-1,0) &= f(-\frac{3}{4})\frac{1}{4} + f(-\frac{1}{2})\frac{1}{4} + f(-\frac{1}{4})\frac{1}{4} + f(0)\frac{1}{4} = \frac{3+2+1}{16} = \frac{3}{8} \end{aligned}$$

The interval from 0 to 1 gives the estimates:

$$\begin{aligned} A_L(0,1) &= f(0)\frac{1}{4} + f(\frac{1}{4})\frac{1}{4} + f(\frac{1}{2})\frac{1}{4} + f(\frac{3}{4})\frac{1}{4} = \frac{3+2+1}{16} = \frac{3}{8} \\ A_R(0,1) &= f(\frac{1}{4})\frac{1}{4} + f(\frac{1}{2})\frac{1}{4} + f(\frac{3}{4})\frac{1}{4} + f(1)\frac{1}{4} = \frac{4+3+2+1}{16} = \frac{5}{8} \end{aligned}$$

To find a lower bound for the area in the given interval, we need to add the two lower bounds together, and similarly for the upper bound. Then the area we require is between $\frac{6}{8}$ and $\frac{10}{8}$. This problem could have been simplified by recognizing that $y = |x|$ is an even function. Then we would only have to double the lower and upper bounds for the area from $x = 0$ to $x = 1$.

Exercises:

1. Using the method described in this section estimate the area under the curve
   (a) $y = x^2$ between $x = 3$ and $x = 6$ using 3 rectangles and finding the upper and lower limits.
   (b) $y = 3x^2 + 1$ between $x = 0$ and $x = 4$ using 8 rectangles and finding the upper and lower limits.
   (c) $y = 4 - x^2$ between $x = -2$ and $x = 0$ using 4 rectangles and finding the upper and lower limits.

Section 2 Integrating Polynomials

Integration is a technique for finding, amoungst other things, the area under curves. Conceptually, it is like the method of left and right rectangles, but the number of subintervals that the interval of interest is broken up into is infinite, so we get an exact area where the lower and upper bounds are equal. We will not give the details of how one takes the limit of an infinite number of subintervals - we will just state some integration results.

Integration involves anti derivatives, so we will first look at these. The anti derivative of a function f is another function F such that

$$f(x) = F'(x)$$

Thus if $f(x)$ is the derivative of $F(x)$ then $F(x)$ is the anti derivative of $f(x)$. Worksheet 3.8 has an introduction to derivatives. Therefore we can reverse the rules that we had for polynomial differentiation to get anti derivative rules. Recall that if $f(x) = ax^n$, then $f'(x) = anx^{n-1}$. So if $g(x) = bx^m$ then an anti derivative $G(x)$ (such that $G'(x) = g(x)$) is given by

$$G(x) = \frac{b}{m+1}x^{m+1} \quad m \neq -1$$

We add one to the power of x then divide by the new power of x. Note that if

$$\begin{aligned} G(x) &= \frac{b}{m+1}x^{m+1} \\ \text{then} \quad G'(x) &= \frac{b}{m+1}(m+1)x^{m+1-1} \\ &= bx^m \\ &= g(x) \end{aligned}$$

which is what is required. Given $f'(x) = 2x$, then we could have $f(x) = x^2+1$ or $f(x) = x^2+3$ or $f(x) = x^2-4$; notice they differ by the constant term. To compensate for this - the property that the derivative of a constant is zero - we add a constant, usually denoted as c, to the anti derivative. We need more information to find distinct values of c.

Example 1 : Find the anti derivative $F(x)$ of the function $f(x) = 2x + 1$. Note $x^0 = 1$.

$$\begin{aligned} F(x) &= \frac{2x^{1+1}}{2} + \frac{1x^{0+1}}{1} + c \\ &= x^2 + x + c \end{aligned}$$

Example 2 : Find the anti derivative $G(x)$ of the function $g(x) = x^2 + 3x$.

$$\begin{aligned} G(x) &= \frac{x^{2+1}}{3} + \frac{3x^{1+1}}{2} + c \\ &= \frac{x^3}{3} + \frac{3x^2}{2} + c \end{aligned}$$

Example 3 : Find the anti derivative $H(x)$ of the function $h(x) = 5x^4 + 3x^2 + x + x^{-5} + 3$.

$$\begin{aligned} H(x) &= \frac{5x^{4+1}}{5} + \frac{3x^{2+1}}{3} + \frac{x^{1+1}}{2} + \frac{x^{-5+1}}{-4} + \frac{3x^{0+1}}{1} + c \\ &= x^5 + x^3 + \frac{x^2}{2} - \frac{x^{-4}}{4} + 3x + c \end{aligned}$$

Example 4 : Find the anti derivative $F(x)$ of $f(x) = x^{-2}$.

$$F(x) = \frac{x^{-2+1}}{-1} = \frac{x^{-1}}{-1} + c = \frac{-1}{x} + c$$

Example 5 : Find the anti derivative $F(x)$ of $f(x) = 1$.

$$F(x) = \frac{1x^{0+1}}{1} = x + c$$

Example 6 : Find the anti derivative of $f(x) = \frac{3}{x^2} + 4x + 5$. Call the anti derivative $F(x)$.

$$\begin{aligned} f(x) &= 3x^{-2} + 4x + 5 \\ F(x) &= \frac{3x^{-1}}{-1} + \frac{4x^2}{2} + 5x + C \\ &= -\frac{3}{x} + 2x^2 + 5x + C \end{aligned}$$

Exercises:

1. Find the anti derivative of each of the following functions

   (a) $6x^2 + 8x - 3$
   (b) $10x^4 - 3x^2 + 5$
   (c) $3x^4 - 6x^2 - 7$
   (d) $x + 3$
   (e) $x^3 - x^{-3} + 2x + 1$
   (f) $\frac{8}{x^3} - \frac{1}{x^2} + 3x + 4$
   (g) $4x^2 - \frac{7}{x^4} + 2$
   (h) $x^4 - 2x$
   (i) $63x^5 - 1$
   (j) $\frac{4}{x^3} - \frac{6}{x^2}$

Section 3 Integration

The area under the curve $y = f(x)$ between $x = a$ and $x = b$, where $f(x) \geq 0$ for $a \leq x \leq b$, is given by the formula

$$A = \int_a^b f(x)\,dx$$

This is read as the integral of the function $f(x)$ from a to b (where a is taken to be the smaller number). The integral can be evaluated using

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$$

where $F(x)$ is an anti derivative of $f(x)$.

This is called a definite integral because we integrate between two given values $x = a$ and $x = b$ to obtain a single value. An indefinite integral is written as

$$\int f(x)\,dx = F(x)$$

where again $F(x)$ is an anti derivative of $f(x)$.

Example 1 : Calculate $\int 3x^2\,dx$.

$$\begin{aligned} \int 3x^2\,dx &= \frac{3x^{2+1}}{3} + c \\ &= \frac{3x^3}{3} + c \\ &= x^3 + c \end{aligned}$$

We have used the fact that $\int 3x^2\,dx = 3\int x^2\,dx$. In other words, we can 'pull' the 3 through the integral sign because the 3 is independent of the variable that we are integrating with respect to, which is x in this case. In general $\int af(x)\,dx = a\int f(x)\,dx$.

Note: An indefinite integral is the same as calculating the anti derivative.

Example 2 : Calculate $\int_0^1 (x+3)\,dx$.

$$\begin{aligned}\int_0^1 (x+3)\,dx &= \left(\frac{x^{1+1}}{2} + \frac{3x^{0+1}}{1}\right)\Big]_0^1 \\ &= \left(\frac{x^2}{2} + 3x\right)\Big]_0^1 \\ &= (\frac{1^2}{2} + 3\times 1) - (\frac{0^2}{2} + 3\times 0) \\ &= 3\frac{1}{2}\end{aligned}$$

Example 3 : Calculate the area under the curve $f(x) = x^2$ between $x = 3$ and $x = 6$. The area is given by

$$\begin{aligned}A = \int_3^6 f(x)\,dx &= \int_3^6 x^2\,dx \\ &= \frac{x^3}{3}\Big]_3^6 \\ &= F(6) - F(3) \\ &= \frac{6^3}{3} - \frac{3^3}{3} \\ &= 63\end{aligned}$$

Recall that, in example 1 in section 1, we found that the area was between 58 and 77.

Example 4 : Calculate the area under $f(x) = x$ between $x = 5$ and $x = 7$.

$$\begin{aligned}A = \int_5^7 x\,dx &= \frac{x^2}{2}\Big]_5^7 \\ &= \frac{49}{2} - \frac{25}{2} \\ &= 12\end{aligned}$$

See the example in section 1 for comparison.

Example 5 : Calculate the area under $f(x) = x^4 + x^2$ between $x = -1$ and $x = 0$.

$$\begin{aligned} A = \int_{-1}^{0} (x^4 + x^2)\, dx &= \left(\frac{x^5}{5} + \frac{x^3}{3}\right)\Bigg]_{-1}^{0} \\ &= \left(\frac{0^5}{5} + \frac{0^3}{3}\right) - \left(\frac{(-1)^5}{5} + \frac{(-1)^3}{3}\right) \\ &= 0 - \left(\frac{-1}{5} - \frac{1}{3}\right) \\ &= \frac{8}{15} \end{aligned}$$

Exercises:

1. Calculate the following integrals

   (a) $\int_{-2}^{3} x + 7\, dx$

   (b) $\int_{1}^{4} x^2 + 6\, dx$

   (c) $\int_{3}^{5} x + 2\, dx$

   (d) $\int_{0}^{4} x^2 + x - 1\, dx$

   (e) $\int_{-1}^{2} 3x + 4\, dx$

   (f) $\int_{0}^{2} 6 - 3x^2\, dx$

   (g) $\int_{1}^{3} x^3 - 2x\, dx$

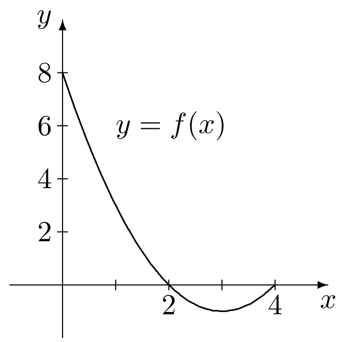
   (h) $\int_{0}^{4} x + 2\, dx$

   (i) $\int_{-3}^{-1} x^3 + x^2 - 6x\, dx$

   (j) $\int_{4}^{6} x + 3\, dx$

Section 4 Integration Continued

As a further investigation of the area under a curve, we will look at the graph of the function $f(x) = x^2 - 6x + 8$.



We will find the area that is shaded. First find the shaded area between $x = 0$ and $x = 2$.

$$\begin{aligned}\int_0^2 x^2 - 6x + 8\,dx &= \left[\frac{x^3}{3} - 3x^2 + 8x\right]_0^2 \\ &= \left(\frac{8}{3} - 12 + 16\right) - (0 - 0 + 0) \\ &= 6\frac{2}{3}\end{aligned}$$

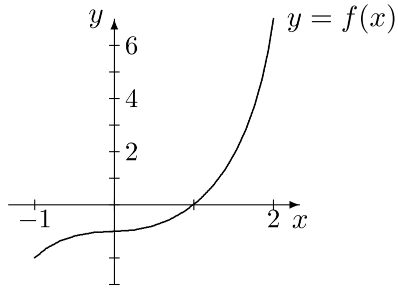
Now see what happens when we use the same method to find the shaded area between $x = 2$ and $x = 4$.

$$\begin{aligned}\int_2^4 x^2 - 6x + 8\,dx &= \left[\frac{x^3}{3} - 3x^2 + 8x\right]_2^4 \\ &= \left(\frac{64}{3} - 48 + 32\right) - \left(\frac{8}{3} - 12 + 16\right) \\ &= -1\frac{1}{3}\end{aligned}$$

An area cannot be negative. The negative sign indicates that the region is below the x axis – in this situation, the actual measure of the area is found by taking the absolute value of the integral. That is, the shaded area between $x = 2$ and $x = 4$ is

$$\left|\int_2^4 x^2 - 6x + 8\,dx\right| = \left|-1\frac{1}{3}\right| = 1\frac{1}{3}$$

Example 1 : Find the area bounded by the curve $y = x^3 - 1$, the x axis, and which lies between the lines $x = 0$ and $x = 1$. First draw the graph.



The required area is below the x axis, so

$$\begin{aligned} A &= \left| \int_0^1 x^3 - 1\, dx \right| \\ &= \left| \left[\frac{x^4}{4} - x \right]_0^1 \right| \\ &= \left| \left(\frac{1}{4} - 1 \right) - \left(\frac{0}{4} - 0 \right) \right| \\ &= \left| -\frac{3}{4} \right| \\ &= \frac{3}{4} \end{aligned}$$

Example 2 : Find the area bound by the curve $y = x^3 - 1$, the x axis, and the lines $x = 0$ and $x = 3$.

Using the graph from the previous example as a guide, we see that the region from $x = 0$ to $x = 1$ is below the axis, and the region from $x = 1$ to $x = 3$ is above the x axis. So the area we want is

$$\begin{aligned} A &= \left| \int_0^1 x^3 - 1\, dx \right| + \int_1^3 x^3 - 1\, dx \\ &= \left| \left[\frac{x^4}{4} - x \right]_0^1 \right| + \left[\frac{x^4}{4} - x \right]_1^3 \\ &= \left| \left(\frac{1}{4} - 1 \right) - \left(\frac{0}{4} - 0 \right) \right| + \left(\frac{81}{4} - 3 \right) - \left(\frac{1}{4} - 1 \right) \\ &= \left| -\frac{3}{4} \right| + 18 \\ &= 18\frac{3}{4} \end{aligned}$$

Exercises for Worksheet 4.2

1. (a) Use the method of left and right rectangles to find upper and lower bounds for the following functions and integration limits:
   i. $y = \sqrt{x}$ between $x = 0$ and $x = 1$ using 5 subdivisions.
   ii. $y = \frac{1}{x}$ between $x = 1$ and $x = 2$ using 10 subdivisions.

   (b) Find the anti derivative of the following functions:

   i. $f(x) = 1 + x + x^2$ ii. $g(x) = x^{\frac{1}{2}}$ iii. $h(x) = \frac{4}{x^3}$

   (c) Evaluate the following definite integrals:

   i. $\displaystyle\int_0^4 7x\,dx$ ii. $\displaystyle\int_0^1 (1 - y^2)\,dy$ iii. $\displaystyle\int_1^2 3t^2\,dt$

2. (a) By using rectangles of width 1, find the area under $y = [x]$ between $x = 0$ and $x = 5$ where $[x]$ is the 'greatest integer' function e.g. $[3.9] = 3$, $[4.1] = 4$.

   (b) Is the function in (i) monotonically increasing?

   (c) Which is greater, $\int_1^2 x\,dx$ or $\int_1^2 \sqrt{x}\,dx$?

   (d) Calculate the area of the region bounded by the graph of $f(x) = (x-2)^2$, the x-axis, and between $x = 2$ and $x = 3$.

   (e) Calculate the area bounded by the curve $y = x^2(3 - x)$ and the x-axis.

   (f) If $\displaystyle\int_{-1}^{a} x\,dx = 0$, evaluate a.

   (g) If $c\displaystyle\int_{-2}^{2} (x - 5)\,dx = 1$, evaluate c.

3. (a) Calculate the area bound by the curves $f(x) = \frac{x^2}{4} - 2$ and $g(x) = x + 1$.
   (Hint: Find the points of intersection of the two curves, and calculate both areas.)

   (b) Show, by integration, that the area of a unit square is:
   (a) Bisected by the line $y = x$.
   (b) Trisected by the curves $y = x^2$ and $y = \sqrt{x}$.

   (c) The marginal revenue, MR, that a manufacturer receives for his goods is given by $MR = \frac{dR}{dq} = 100 - 0.03q$. Find the total revenue function $R(q)$.

   (d) The density curve of a 10-metre beam is given by $\rho(x) = 3x + 2x^2 - x^{\frac{3}{2}}$ where x is the distance measured from one edge of the beam. The mass of the beam is calculated to be the area under the curve $\rho(x)$ between 0 and x. Find the mass of the beam.

Worksheet 4.3 Integrating Special Functions

Section 1 Exponential and Logarithmic Functions

Recall from worksheet 3.10 that the derivative of e^x is e^x. It then follows that the anti derivative of e^x is e^x:

$$\int e^x\,dx = e^x + c$$

In worksheet 3.10 we also discussed the derivative of $e^{f(x)}$ which is $f'(x)e^{f(x)}$. It then follows that

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

where $f(x)$ can be any function. There are other ways of doing such integrations, one of which is by substitution.

Example 1 : Evaluate the indefinite integral $\int 3e^{3x+2}\,dx$.

We recognize that $3 = \frac{d(3x+2)}{dx}$ so that the expression we are integrating has the form $f'(x)e^{f(x)}$. Then

$$\int 3e^{3x+2}\,dx = e^{3x+2} + c$$

Alternatively, we could do it by substitution: let $u = 3x+2$. Then $du = 3dx$, and

$$\int 3e^{3x+2}\,dx = \int e^u\,du = e^u = e^{3x+2}$$

Note that the integral of the function e^{ax+b} (where a and b are constants) is given by

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

Example 2 : Find the area under the curve $y = e^{5x}$ between 0 and 2.

$$\begin{aligned} A &= \int_0^2 e^{5x}\,dx \\ &= \frac{1}{5}e^{5x}\Big]_0^2 \\ &= \frac{1}{5}e^{10} - \frac{1}{5}e^0 \\ &= \frac{1}{5}(e^{10} - 1) \end{aligned}$$

We used the property that for any real number x, $x^0 = 1$.

Recall that the derivative of $\log_e x$ is $\frac{1}{x}$. Then the anti derivative of $\frac{1}{x}$ is $\log_e x$. Notice that $\frac{1}{x} = x^{-1}$, and that if we had used the rules we have developed to find the anti derivatives of things like x^m, we would have the anti derivative of x^{-1} being $\frac{x^{-1+1}}{-1+1} = \frac{x^0}{0}$ which is not defined as we can not divide by zero. So we have the special rule for the anti derivative of $1/x$:

$$\int \frac{1}{x}\,dx = \log_e x + c$$

Recall that the derivative of $\log_e f(x)$ is $\frac{f'(x)}{f(x)}$. Then we have

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x) + c$$

<u>Example 3</u> : Evaluate the indefinite integral $\int \frac{5}{5x+2}\,dx$. This has the form $\int \frac{f'(x)}{f(x)}\,dx$ so we get

$$\int \frac{5}{5x+2}\,dx = \log_e(5x+2) + c$$

Note that when you need to integrate a function like $1/(ax+b)$ (where a and b are constants), then

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\int \frac{a}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + c$$

<u>Example 4</u> : Find the area under the curve $f(x) = 1/(2x+3)$ between 3 and 11.

$$\begin{aligned}
A &= \int_3^{11} \frac{1}{2x+3}\,dx \\
&= \frac{1}{2}\log_e(2x+3)\Big]_3^{11} \\
&= \frac{1}{2}\log_e(2\times 11+3) - \frac{1}{2}\log_e(2\times 3+3) \\
&= \frac{1}{2}\log_e 25 - \frac{1}{2}\log_e 9 \\
&= \log_e(25)^{\frac{1}{2}} - \log_e(9)^{\frac{1}{2}} \\
&= \log_e \frac{5}{3}
\end{aligned}$$

Section 2 Integrating Trig Functions

To integrate trig functions we need to recall the derivatives of trig functions. We can then work out the anti derivatives of $\cos x$, $\sin x$, and $\sec^2 x$. For more complicated integrals we need special techniques that you will learn in first-year maths. The derivatives of the trig functions are:

$$\begin{aligned} g(x) &= \sin(ax+b) & g'(x) &= a\cos(ax+b) \\ f(x) &= \cos(ax+b) & f'(x) &= -a\sin(ax+b) \\ h(x) &= \tan(ax+b) & h'(x) &= a\sec^2(ax+b) \end{aligned}$$

Example 1 : Evaluate the indefinite integral $\int \sin 3x\, dx$.

$$\int \sin 3x\, dx = \frac{-1}{3}\cos 3x + c$$

Note : A good way of checking your answers to indefinite integrals is to differentiate them. You should recover the function that you started with.

Example 2 : Find the area under the curve $y = \cos x$ between 0 and $\frac{\pi}{2}$.

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \cos x\, dx \\ &= \sin x\Big]_0^{\frac{\pi}{2}} \\ &= \sin\frac{\pi}{2} - \sin 0 \\ &= 1 \text{ square units} \end{aligned}$$

Example 3 : Find $\int f(x)\, dx$ if $f(x) = -3\sin(3x+2)$.

$$\int -3\sin(3x+2)\, dx = \cos(3x+2) + c$$

Example 4 : What is the area under the curve $y = \sec^2 \frac{x}{2}$ between $\frac{\pi}{2}$ and 0?

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2}\, dx \\ &= \frac{1}{1/2} \tan \frac{x}{2} \Big]_0^{\frac{\pi}{2}} \\ &= 2 \tan \frac{x}{2} \Big]_0^{\frac{\pi}{2}} \\ &= 2 \tan \frac{\pi}{4} - 2 \tan 0 \\ &= 2 - 0 \\ &= 2 \quad \text{square units} \end{aligned}$$

Example 5 : Evaluate the indefinite integral $\int 5 \sec^2 5x\, dx$.

$$\int 5 \sec^2 5x\, dx = \tan 5x + c$$

Exercises for Worksheet 4.3

1. (a) Find the anti derivative of

   i. e^{-4x}
   ii. $\sqrt{e^x}$
   iii. $\dfrac{7-6x}{8+7x-3x^2}$
   iv. $\cos 2x$
   v. $\sec^2(5x-2)$
   vi. $\dfrac{1-x}{x^2}$

   (b) Evaluate

   i. $\displaystyle\int_0^{\frac{1}{2}} e^{2x}\,dx$
   ii. $\displaystyle\int_{-1}^{1} \frac{2x+1}{x^2+x+1}\,dx$
   iii. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$
   iv. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\cos x\,dx$

2. (a) Calculate the area under the curve $y = \frac{2}{x+3}$ from $x = 2$ to $x = 3$.

   (b) Calculate the area under the curve $y = e^{3x}$ from $x = 0$ to $x = 3$.

   (c) The area under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = b$ is 1 unit. What is b?

   (d) Find the points of intersection of the curve $y = \sin x$ with the line $y = \frac{1}{2}$ and hence find the area between the two curves (from one intersection to the next). There are two possible areas you can end up with; choose the one above $y = \frac{1}{2}$.

   (e) Show, by simple division, that $\dfrac{x+6}{x+2} = 1 + \dfrac{4}{x+2}$. Hence evaluate $\displaystyle\int \frac{x+6}{x+2}\,dx$.

Worksheet 4.4 Applications of Integration

Section 1 Movement

Recall that the derivative of a function tells us about its slope. What does the slope represent? It is the change in one variable with respect to the other variable. Say a line has a constant slope of 4; then for every 1 unit change in x, there will be a 4 unit change in y. Say we had a function that represented the movement of a car, so that the distance was plotted as a function of time. The change in distance over a small amount of time would represent the speed of the car. Thus in this case the slope of the function represents the speed of the car, and is given by $\frac{dx}{dt}$. The rate of change in speed of the car is called acceleration and this is given by $\frac{d}{dt}(\frac{dx}{dt}) = \frac{d^2x}{dt^2}$.

So if we have a function $x = f(t)$ that represents distance as a function of time, then $\frac{dx}{dt}$ is the speed and $\frac{d^2x}{dt^2}$ is the acceleration. Conversely, if we have a function that represents the velocity of a vehicle and we integrate it we get the distance travelled as a function of time.

The methods of integration and differentiation can be used to solve problems involving movement.

Example 1 : If the velocity of a particle is given by $v = 3t^2$, what is the distance travelled as a function of time? Since $v = \frac{dx}{dt}$, where x is the distance, the anti derivative of v will tell us the distance.

$$x = \frac{3t^3}{3} + c = t^3 + c$$

Example 2 : If the velocity of a particle is given as $v = 3t^2$ metres per second, what is the distance travelled between $t = 0$ and $t = 2$? In example 1, we worked out that the distance travelled was $x = t^3 + c$. Therefore

$$\text{Distance } = (t^3 + c)\Big]_0^2 = (2^3 + c) - (0^3 + c) = 8 \text{ metres}$$

The particle covers a distance of 8 metres between $t = 0$ and $t = 2$.

Section 2 Initial-Value Problems

Recall that, when working out anti derivative problems, there is a constant of integration that is undetermined (and which we have usually denoted by c). Initial-value problems ask us to find anti derivatives which take on specific values at certain points so that we can determine the value of the constant.

Example 1 : If $\frac{dx}{dt} = 5$ and $x = 9$ when $t = 0$, what is x as a function of time? As $\frac{dx}{dt} = 5$, then $x = 5t + c$. Using the information that when $t = 0$, $x = 9$ we can now write an equation to solve for c:

$$9 = 5 \times 0 + c$$

which has the solution $c = 9$. The complete solution for x is $x = 5t + 9$.

Example 2 : The acceleration of a car is given by $a = 6t$ and the velocity is 2 when $t = 0$, and the distance from home is 1 when $t = 0$. What is the distance from home as a function of time? Acceleration is the derivative of velocity, so velocity is the anti derivative of acceleration. Then

$$v = \int 6t\,dt = 3t^2 + c_1$$

When $t = 0$, $v = 2$, so that we can write down an equation for c_1 and solve it: $2 = 0 + c_1$. Therefore, $c_1 = 2$. The velocity is then $v = 3t^2 + 2$. Distance is the anti derivative of velocity, which gives

$$x = \int (3t^2 + 2)\,dt = t^3 + 2t + c_2$$

Using $x = 1$ when $t = 0$, we can write down an equation for c_2: $1 = 0 + 0 + c_2$. Therefore $c_2 = 1$, and so the distance as a function of time is

$$x = t^3 + 2t + 1$$

Example 3 : If $\frac{d^2x}{dt^2} = 3$, and when $t = 0$ we have $\frac{dx}{dt} = 0$ and $x = 0$, what is x as a function of t?

$$\begin{aligned} \frac{d^2x}{dt^2} &= 3 \\ \frac{dx}{dt} &= 3t + c \end{aligned}$$

Using the information we are given for $\frac{dx}{dt}$ at $t = 0$, we find $3 \times 0 + c = 0$ so that $c = 0$. Then $\frac{dx}{dt} = 3t$ for all t. The anti derivative of this will give us x:

$$x = \frac{3t^2}{2} + c$$

Using the information for x at $t = 0$, we find $c = 0$, so that $x = \frac{3t^2}{2}$ for all t.

Section 3 Application to Growth

Exponential functions are used to represent the growth and decay of populations and radioactive elements, among other things. We can use a general form of an equation for exponential growth or decay and we find a specific equation which uses initial values as in the application of integration to motion.

Exponential growth and decay is represented by the equation $P(t) = P(0)e^{kt}$ where $P(t)$ is the population at time t, $P(0)$ is the population at $t = 0$, and k is some constant which depends on the population being looked at. A similar formula applies to the decay of radioactive material. $P(0)$ would then represent the amount of radioactive material at $t = 0$.

Example 1 : What is $P(10)$ if $P(0) = 100$ and $k = 1$ given $P(t) = P(0)e^{kt}$.

$$\begin{aligned} P(10) &= P(0)e^{kt} \\ &= 100e^{10} \end{aligned}$$

Example 2 : If $P(10) = 1000$ and $P(0) = 100$, what is k in the expression $P(t) = P(0)e^{kt}$? We put $t = 10$ into the equation for $P(t)$ and equate this to what we are given at $P(10)$.

$$P(10) = 1000 = 100e^{k10}$$

This equation can be solved for k:

$$\begin{aligned} 1000 &= 100e^{10k} \\ 10 &= e^{10k} \\ \log 10 &= 10k \\ k &= \frac{1}{10}\log 10 \end{aligned}$$

Example 3 : If the growth constant for a population of bees is $\frac{1}{10}$ and the initial population of a hive is 75, what is the population at time t?

$$\begin{aligned} P(t) &= P(0)e^{kt} \\ &= 75e^{\frac{1}{10}t} \end{aligned}$$

Notice that if $k > 0$ the population is growing but if $k < 0$ the population is getting smaller.

Example 4 : For what values of k does the population $P(t) = P(0)e^{kt}$ remain constant? We need $P(t) = P(0)$ for all t. Then

$$\begin{aligned} P(t) &= P(0)e^{kt} \\ 1 &= e^{kt} \end{aligned}$$

This is true when $k = 0$.

Exercises for Worksheet 4.4

1. The derivatives of a function and one point on its graph are given. Find the function.

   (a) $\dfrac{dy}{dx} = x^3 + x^2 - 3$; $(1, 5)$

   (b) $\dfrac{dy}{dx} = 2x(x+1)$; $(2, 0)$

   (c) $y' = \cos x$; $(\frac{\pi}{6}, 4)$

   (d) $y' = \dfrac{x}{\sqrt{10 - x^2}}$; $(1, 5)$

2. (a) Find $f(x)$ if its gradient function is $2x - 2$ and $f(1) = 4$.

   (b) The velocity $v(t)$ of a particle moving in a straight line is given by $v(t) = 12t^2 - 6t + 1$, $t \geq 0$. Find its position coordinate $s(t) = \int v(t)\, dt$ given that $s(1) = 4$.

   (c) If $\frac{dx}{dt} = kx$ and $x = 10$ when $t = 0$,

      i. Show that $x = 10e^{kt}$.

      ii. Find k if $x = 20$ when $t = 10$.

   (d) A radioactive substance decays according to the rule $\frac{dM}{dt} = -0.2M$. If $M = 5$ when $t = 0$,

      i. Show that $M = 5e^{-0.2t}$.

      ii. Find M when $t = 5$.

   (e) A ship travelling at 10 metres per second is subjected to water resistance proportional to the speed. The engines are cut and the ship slows down according to the rule $\frac{dv}{dt} = -kv$.

      i. Show that the velocity after t seconds is given by $v = 10e^{-kt}$ metres per second.

      ii. If, after 20 seconds, $v = 5$m/s, find k.

3. (a) A particle moves with constant acceleration of 5.8 metres/second squared. It starts with an initial velocity of 0.2m/s, and an initial position of 25m. Find the equation of motion of the particle given $\int$ (acceleration) dt = velocity, and $\int$ (velocity) dt = position.

   (b) If the instantaneous rate of change of a population is $50t^2 - 100t^{\frac{3}{2}}$ (measured in individuals per year) and the initial population is 25000 then

      (a) What is the population after t years?

      (b) What is the population after 25 years?

   (c) A particle moves along a straight line with an acceleration of $a = 4 \sin \frac{\pi t}{2}$ m/s^2. If the displacement at $t = 0$ is 0, and the initial velocity is $-\frac{8}{\pi}$m/s, find

      i. The acceleration after 2 seconds.

      ii. The velocity after 2 seconds.

      iii. The displacement after 2 seconds.

Worksheet 4.5 Polynomials

Section 1 Introduction to polynomials

A polynomial is an expression of the form

$$p(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n, \qquad (n \in \mathbb{N})$$

where $p_0, p_1, \ldots, p_n$ are constants and x os a variable.

Example 1 : $f(x) = 3 + 5x + 7x^2 - x^4 \qquad (p_0 = 3, p_1 = 5, p_2 = 7, p_3 = 0, p_4 = -1)$

Example 2 : $g(x) = 2x^3 + 3x \qquad (p_0 = 0, p_1 = 3, p_2 = 0, p_3 = 2)$

- The constants $p_0, \ldots, p_1$ are called coefficients. In Example 1 the coefficient of x is 5 and the coefficient of x^4 is -1. The term which is independent of x is called the constant term. In Example 1 the constant term of $f(x)$ is 3; in Example 2 the constant term of $g(x)$ is 0.
- A polynomial $p_0 + p_1x + \cdots + p_nx^n$ is said to have degree n, denoted deg n, if $p_n \neq 0$ and x^n is the highest power of x which appears. In Example 1 the degree of $f(x)$ is 4; in Example 2 the degree of $g(x)$ is 3.
- A zero polynomial is a polynomial whose coefficients are all 0, i.e. $p_0 = p_1 = \cdots = p_n = 0$.
- Two polynomials are equal if all the coefficients of the corresponding powers of x are equal.

Exercises:

1. Find (i) the constant term, (ii) the coefficient of x^4 and (iii) the degree of the following polynomials.
   (a) $x^4 + x^3 + x^2 + x + 1$
   (b) $9 - 3x^2 + 7x^3$
   (c) $x - 2$
   (d) $10x^5 - 3x^4 + 5x + 6$
   (e) $3x^6 + 7x^4 + 2x$
   (f) $7 + 5x + x^2 - 6x^4$
2. Suppose $f(x) = 2ax^3 - 3x^2 - b^2x - 7$ and $g(x) = cx^4 + 10x^3 - (d + 1)x^2 - 4x + e$. Find values for constants a, b, c, d and e given that $f(x) = g(x)$.

Section 2 Operations on polynomials

Suppose

$$h(x) = 3x^2 + 4x + 5 \qquad \text{and} \qquad k(x) = 7x^2 - 3x.$$

- *Addition/Subtraction*: To add/subtract polynomials we combine like terms.

  Example 1 : We have

  $$h(x) + k(x) = 10x^2 + x + 5$$

- *Multiplication*: To multiply polynomials we expand and then simplify their product.

  Example 2 : We have

  $$\begin{aligned} h(x) \cdot k(x) &= (3x^2 + 4x + 5)(7x^2 - 3x) \\ &= 21x^4 - 9x^3 + 28x^3 - 12x^2 + 35x^2 - 15x \\ &= 21x^4 + 19x^3 + 23x^2 - 15x \end{aligned}$$

- *Substitution*: We can substitute in different values of x to find the value of our polynomial at this point.

  Example 3 : We have

  $$\begin{aligned} h(1) &= 3(1)^2 + 4(1) + 5 = 12 \\ k(-2) &= 7(-2)^2 - 3(-2) = 34 \end{aligned}$$

- *Division*: To divide one polynomial by another we use the method of long division.

  Example 4 : Suppose we wanted to divide $3x^3 - 2x^2 + 4x + 7$ by $x^2 + 2x$.

  $$\begin{array}{r|rrrr} & 3x & -8 & & \\ \hline x^2 + 2x & 3x^3 & -2x^2 & +4x & +7 \\ & 3x^3 & +6x^2 & & \\ \cline{2-4} & & -8x^2 & +4x & \\ & & -8x^2 & -16x & \\ \cline{3-5} & & & 20x & +7 \end{array}$$

  So $3x^3 - 2x^2 + 4x + 7 = (x^2 + 2x)(3x - 8) + (20x + 7)$.

More formally, suppose $p(x)$ and $f(x)$ are polynomials where $\deg p(x) \geq \deg f(x)$. Then dividing $p(x)$ by $f(x)$ gives us the identity

$$p(x) = f(x)q(x) + r(x),$$

where $q(x)$ is the quotient, $r(x)$ is the remainder and $\deg r(x) < \deg f(x)$.

Example 5 : Dividing $p(x) = x^3 - 7x^2 + 4$ by $f(x) = x - 1$ we obtain the following result:

$$\begin{array}{r|rrrr}
 & x^2 & -6x & -6 & \\
\hline
x-1 & x^3 & -7x^2 & +0x & +4 \\
 & x^3 & -x^2 & & \\
\cline{2-3}
 & & -6x^2 & +0x & \\
 & & -6x^2 & +6x & \\
\cline{3-5}
 & & & -6x & +4 \\
 & & & -6x & +6 \\
\cline{4-5}
 & & & & -2
\end{array}$$

Here the quotient is $q(x) = x^2 - 6x - 6$ and the remainder is $r = -2$. *Note:* As we can see, division doesn't always produce a polynomial answer – sometimes there's just a constant remainder.

Exercises:

1. Perform the following operations and find the degree of the result.
   (a) $(2x - 4x^2 + 7) + (3x^2 - 12x - 7)$
   (b) $(x^2 + 3x)(4x^3 - 3x - 1)$
   (c) $(x^2 + 2x + 1)^2$
   (d) $(5x^4 - 7x^3 + 2x + 1) - (6x^4 + 8x^3 - 2x - 3)$
2. Let $p(x) = 3x^4 + 7x^2 - 10x + 4$. Find $p(1)$, $p(0)$ and $p(-2)$.
3. Carry out the following divisions and write your answer in the form $p(x) = f(x)q(x) + r(x)$.
   (a) $(3x^3 - x^2 + 4x + 7) \div (x + 2)$
   (b) $(3x^3 - x^2 + 4x + 7) \div (x^2 + 2)$
   (c) $(x^4 - 3x^2 - 2x + 4) \div (x - 1)$
   (d) $(5x^4 + 30x^3 - 6x^2 + 8x) \div (x^2 - 3x + 1)$

(e) $(3x^4 + x) \div (x^2 + 4x)$

4. Find the quotient and remainder of the following divisions.

   (a) $(2x^4 - 2x^2 - 1) \div (2x^3 - x - 1)$
   (b) $(x^3 + 2x^2 - 5x - 3) \div (x + 1)(x - 2)$
   (c) $(5x^4 - 3x^2 + 2x + 1) \div (x^2 - 2)$
   (d) $(x^4 - x^2 - x) \div (x + 2)^2$
   (e) $(x^4 + 1) \div (x + 1)$

Section 3 Remainder theorem

We have seen in Section 2 that if a polynomial $p(x)$ is divided by polynomial $f(x)$, where $\deg\, p(x) \geq \deg\, f(x)$, we obtain the expression $p(x) = f(x)q(x) + r(x)$, where $q(x)$ is the quotient, $r(x)$ is the remainder and $\deg\, r(x) < \deg\, f(x)$, or $r = 0$.

Now suppose $f(x) = x - a$, where $a \in \mathbb{R}$. Then

$$\begin{aligned} & p(x) = (x - a)\, q(x) + r(x) \\ \text{i.e.} \quad & p(x) = (x - a)\, q(x) + r, \qquad \text{since} \quad \deg r < \deg f. \end{aligned}$$

So the remainder when $p(x)$ is divided by $x - a$ is $p(a)$. This important result is known as the remainder theorem.

Remainder Theorem: *If a polynomial $p(x)$ is divided by $(x-a)$, then the remainder is $p(a)$.*

Example 1 : Find the remainder when $x^3 - 7x^2 + 4$ is divided by $x - 1$.

Instead of going through the long division process to find the remainder, we can now use the remainder theorem. The remainder when $p(x) = x^3 - 7x^2 + 4$ is divided by $x - 1$ is

$$p(1) = (1)^3 - 7(1)^2 + 4 = -2.$$

Note: Checking this using long division will give the same remainder of -2 see Example 5 from Section 2).

Exercises:

1. Use the remainder theorem to find the remainder of the following divisions and then check your answers by long division.
   (a) $(4x^3 - x^2 + 2x + 1) \div (x - 5)$
   (b) $(3x^2 + 12x + 1) \div (x - 1)$

2. Use the remainder theorem to find the remainder of the following divisions.
   (a) $(x^3 - 5x + 6) \div (x - 3)$
   (b) $(3x^4 - 5x^2 - 20x - 8) \div (x + 1)$
   (c) $(x^4 - 7x^3 + x^2 - x - 1) \div (x + 2)$
   (d) $(2x^3 - 2x^2 + 3x - 2) \div (x - 2)$

Section 4 The factor theorem and roots of polynomials

The remainder theorem told us that if $p(x)$ is divided by $(x - a)$ then the remainder is $p(a)$. Notice that if the remainder $p(a) = 0$ then $(x - a)$ fully divides into $p(x)$, i.e. $(x - a)$ is a factor of $p(x)$. This is known as the factor theorem.

Factor Theorem: *Suppose $p(x)$ is a polynomial and $p(a) = 0$. Then $(x - a)$ is a factor of $p(x)$ and we can write $p(x) = (x - a)q(x)$ for some polynomial $q(x)$.*

Note: If $p(a) = 0$ we call $x = a$ a root of $p(x)$.

We can use trial and error to find solutions of a polynomial $p(x)$ by finding a number a where $p(a) = 0$. If we can find such a number a then we know $(x - a)$ is a factor of $p(x)$, and then we can use long division to find the remaining factors of $p(x)$.

Example 1 :

a) Find all the factors of $p(x) = 6x^3 - 17x^2 + 11x - 2$.
b) Hence find all the solutions to $6x^3 - 17x^2 + 11x - 2 = 0$.

Solution a). By trial and error notice that

$$p(2) = 48 - 66 + 22 - 2 = 0$$

i.e. 2 is a root of $p(x)$.
So $(x-2)$ is a factor of $p(x)$.

To find all the other factors we'll divide $p(x)$ by $(x-2)$.

$$\begin{array}{r|rrrr} & 6x^2 & -5x & +1 & \\ \hline x-2 & 6x^3 & -17x^2 & +11x & -2 \\ & 6x^3 & -12x^2 & & \\ \cline{2-3} & & -5x^2 & +11x & \\ & & -5x^2 & +10x & \\ \cline{3-5} & & & x & -2 \\ & & & x & -2 \\ \cline{4-5} & & & & 0 \end{array}$$

So $p(x) = (x-2)(6x^2-5x+1)$. Now notice

$$\begin{aligned} 6x^2-5x+1 &= 6x^2-3x-2x+1 \\ &= 3x(2x-1)-(2x-1) \\ &= (3x-1)(2x-1) \end{aligned}$$

So $p(x) = (x-2)(3x-1)(2x-1)$ and its factors are $(x-2)$, $(3x-1)$ and $(2x-1)$.

Solution b). The solutions to $p(x) = 0$ occur when

$$x-2=0, \qquad 3x-1=0, \qquad 2x-1=0.$$

That is

$$x=2, \qquad x=\frac{1}{3}, \qquad x=\frac{1}{2}.$$

Exercises:

1. For each of the following polynomials find (i) its factors; (ii) its roots.
   (a) x^3-3x^2+5x-6
   (b) x^3+3x^2-9x+5
   (c) $6x^3-x^2-2x$
   (d) $4x^3-7x^2-14x-3$

2. Given that $x-2$ is a factor of the polynomial $x^3-kx^2-24x+28$, find k and the roots of this polynomial.

3. Find the quadratic whose roots are -1 and $\frac{1}{3}$ nd whose value at $x=2$ is 10.

4. Find the polynomial of degree 3 which has a root at -1, a double root at 3 and whose value at $x = 2$ is 12.

5. (a) Explain why the polynomial $p(x) = 3x^3 + 11x^2 + 8x - 4$ has at least one root in the interval from $x = 0$ to $x = 1$.

   (b) Find all roots of this polynomial.

Exercises for Worksheet 4.5

1. Find the quotient and remainder of the following divisions.
   (a) $(x^3 - x^2 + 8x - 5) \div (x^2 - 7)$
   (b) $(x^3 + 5x^2 + 15) \div (x + 3)$
   (c) $(2x^3 - 6x^2 - x + 6) \div (x - 3)$
   (d) $(x^4 + 3x^3 - x^2 - 2x - 7) \div (x^2 + 3x + 1)$

2. Find the factors of the following polynomials.
   (a) $3x^3 - 8x^2 - 5x + 6$
   (b) $x^3 - 4x^2 + 6x - 4$
   (c) $2x^3 + 5x^2 - 3x$
   (d) $x^3 + 6x^2 + 12x + 8$

3. Solve the following equations.
   (a) $x^3 - 3x^2 + x + 2 = 0$
   (b) $5x^3 + 23x^2 + 10x - 8 = 0$
   (c) $x^3 - 8x^2 + 21x - 18 = 0$
   (d) $x^3 - 2x^2 + 5x - 4 = 0$
   (e) $x^3 + 5x^2 - 4x - 20 = 0$

4. Consider the polynomial $p(x) = x^3 - 4x^2 + ax - 3$.
   (a) Find a if, when $p(x)$ is divided by $x + 1$, the remainder is -12.
   (b) Find all the factors of $p(x)$.

5. Consider the polynomial $h(x) = 3x^3 - kx^2 - 6x + 8$.
   (a) Given that $x - 4$ is a factor of $h(x)$, find k and find the other factors of $h(x)$.
   (b) Hence find all the roots of $h(x)$.

6. Find the quadratic whose roots are -3 and $\frac{1}{5}$ and whose value at $x = 0$ is -3.

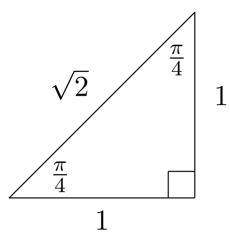
7. Find the quadratic which has a remainder of -6 when divided by $x - 1$, a remainder of -4 when divided by $x - 3$ and no remainder when divided by $x + 1$.

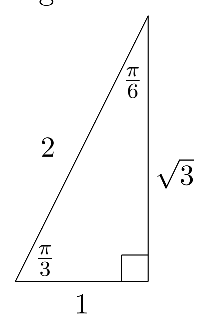
8. Find the polynomial of degree 3 which has roots at $x = 1$, $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$, and whose value at $x = 2$ is -2.

Worksheet 4.6 Properties of Trigonometric functions

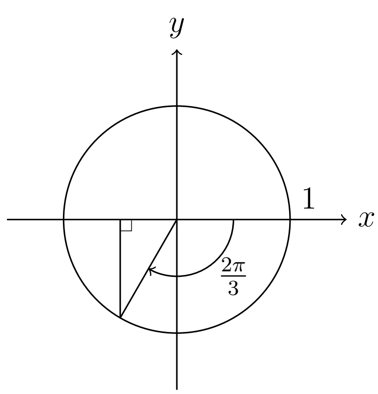
Section 1 Review of Trigonometry

This section reviews some of the material covered in Worksheets 2.2, 3.3 and 3.4. The reader should be familiar with the trig ratios, using radians and working with exact values which arise from the following standard triangles.



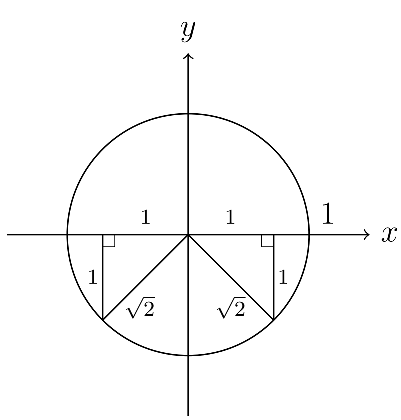


Example 1 : Find the exact value of $\tan \frac{-2\pi}{3}$.



- $-\frac{2\pi}{3}$ lies in the third quadrant and the angle made with the horizontal axis is $\frac{\pi}{3}$.
- tan is positive in the 3rd quadrant
- looking at the corresponding standard triangle in the third quadrant we see that $\tan(\frac{-2\pi}{3}) = +\sqrt{3}$

Example 2 : Find θ if $\sin\theta = -\frac{1}{\sqrt{2}}$ and $0 \leq \theta \leq 2\pi$.



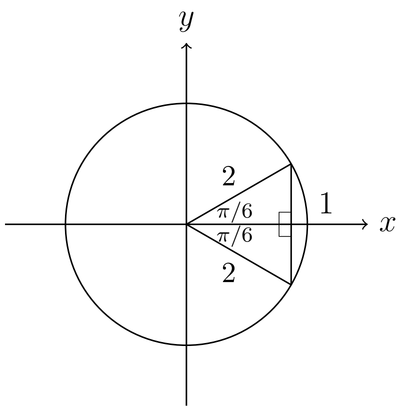
- since $\sin\theta$ is negative it must lie in the third quadrant o fourth quadrant
- looking at the standard triangle where $\sin\theta = \frac{1}{\sqrt{2}}$ in the third and fourth quadrant we see that

$$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

or

$$\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

Example 3 : Find θ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $-\pi \leq \theta \leq \pi$.



- since $\cos\theta$ is positive it must lie in the first or fourth quadrant
- look at the standard triangles where $\cos\theta = \frac{\sqrt{3}}{2}$ in the first and fourth quadrants
- note that $-\pi \leq \theta \leq \pi$
- so $\theta = \frac{\pi}{6}$ or $\theta = -\frac{\pi}{6}$

Exercises:

1. Find the exact values of the following trig ratios.

   (a) $\tan(\frac{5\pi}{3})$

   (b) $\sin(-\frac{10\pi}{3})$

   (c) $\cos(\frac{9\pi}{4})$

   (d) $\sin(\frac{34\pi}{6})$

   (e) $\sec(\frac{5\pi}{6})$

   (f) $\cot(-\frac{11\pi}{4})$

2. Find the value of θ in the following exercises.

   (a) $\cos\theta = -\frac{\sqrt{3}}{2}$ where $0 \leq \theta \leq 2\pi$

   (b) $\tan\theta = \frac{1}{\sqrt{3}}$ where $0 \leq \theta \leq 2\pi$

   (c) $\sin\theta = -\frac{\sqrt{3}}{2}$ where $-\pi \leq \theta \leq \pi$

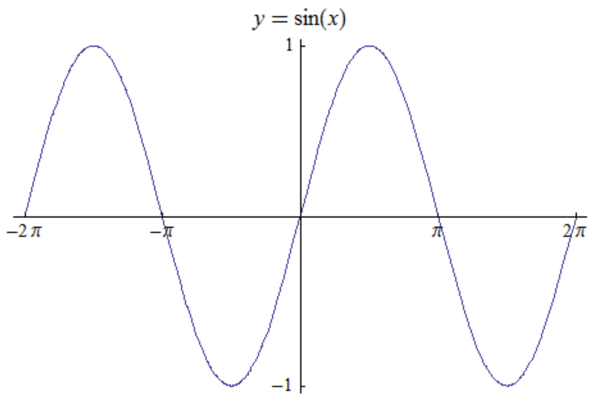
   (d) $\sec\theta = -\sqrt{2}$ where $0 \leq \theta \leq 4\pi$

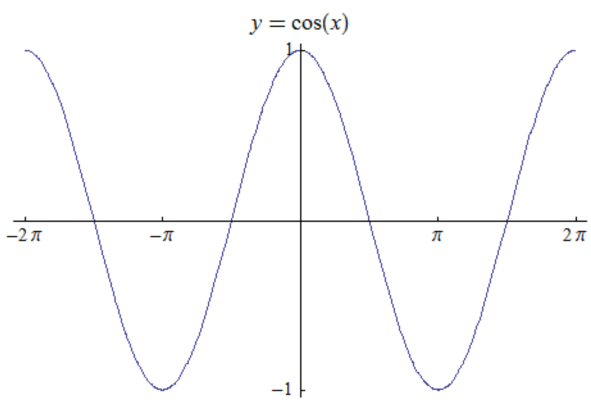
   (e) $\csc\theta = 2$ where $\frac{\pi}{2} \leq \theta \leq 2\pi$

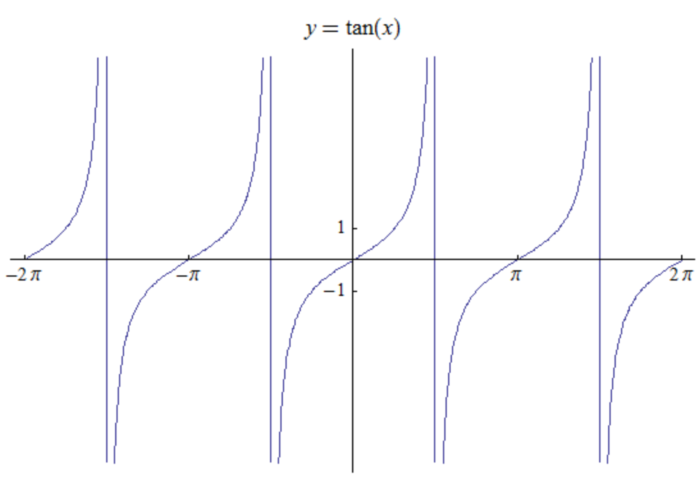
   (f) $\tan^2\theta = 1$ where $0 \leq \theta \leq 2\pi$

Section 2 Graphs of trigonometric functions

Recall the graphs of the trig functions described below for $-2\pi \leq x \leq 2\pi$.







We can see some properties of the trig functions from their graphs.

① These trig functions are periodic – they repeat themselves after a certain period.

- $\sin x$ and $\cos x$ both have period 2π. i.e.

$$\sin x = \sin(x + 2\pi) \qquad \forall\, x \in \mathbb{R}$$
$$\cos x = \cos(x + 2\pi) \qquad \forall\, x \in \mathbb{R}$$

- $\tan x$ has period π. i.e.

$$\tan x = \tan(x + \pi) \qquad \forall\, x \in \mathbb{R}$$

② Note that $\sin x$ and $\cos x$ both lie between -1 and 1.

③ Note that $\tan x$ is undefined for $x = \frac{(2k-1)\pi}{2}$ when $k \in \mathbb{Z}$.

④ From the graphs we can see that

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

⑤ Since $\sin x$ and $\tan x$ are odd functions we have

$$\sin(-x) = -\sin x \qquad \forall\, x \in \mathbb{R}$$
$$\tan(-x) = -\tan x \qquad \forall\, x \in \mathbb{R}$$

Since $\cos x$ is an even function we have

$$\cos(-x) = \cos x \qquad \forall\, x \in \mathbb{R}$$

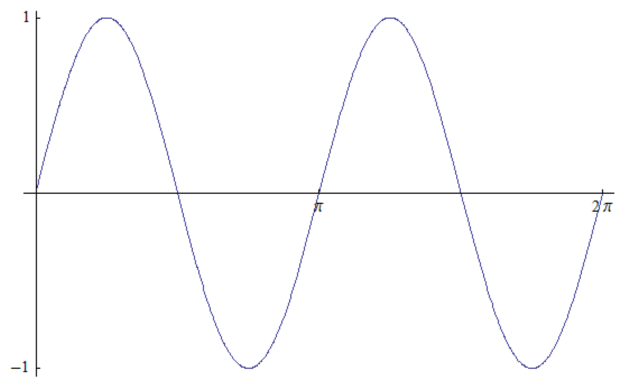
These graphs alter if we change either the period or amplitude, or if there is a phase shift. Consider the graph of $y = \sin x$. In general we can think of this as $y = A \sin n(x - a)$, where

- A is the amplitude
- n alters the period (period $= \frac{2\pi}{n}$)
- by subtracting a from x, the graph shifts to the right by a.

So $y = \sin x$ has amplitude 1 and period 2π.

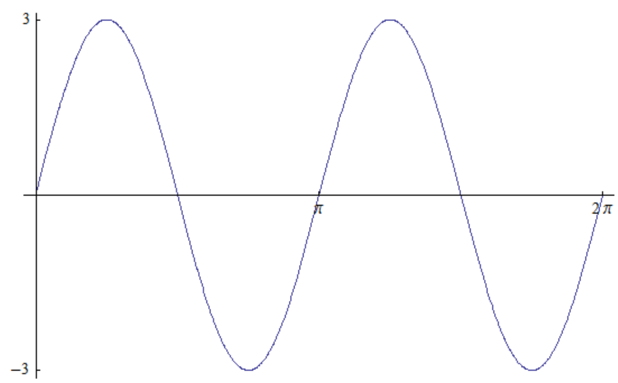
Example 1 : Sketch $y = \sin 2x$ for $0 \le x \le 2\pi$.

Here the period is now $\frac{2\pi}{2} = \pi$.



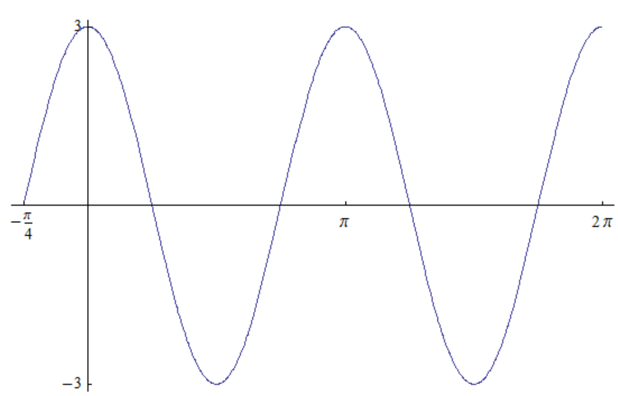
Example 2 : Sketch $y = 3\sin 2x$ for $0 \le x \le 2\pi$.

Here the period is still π but the amplitude is now 3.



Example 3 : Sketch $y = 3\sin 2(x + \frac{\pi}{4})$ for $-\frac{\pi}{4} \le x \le 2\pi$.

The period is π, the amplitude is 3 and there is a phase shift. The graph shifts to the left by $\frac{\pi}{4}$.



Exercises:

1. Sketch the following graphs and state the period for each.

   (a) $y = \cos 4x$

   (b) $y = 2\cos 4(x - \frac{\pi}{3})$

   (c) $y = \tan 2(x + \frac{\pi}{2})$

   (d) $y = 1 + \sin(\frac{x-\pi}{3})$

   (e) $y = 2 - \cos(x + \frac{\pi}{6})$

   (f) $y = |\cos x|$

   (g) $y = \cos |x|$

2. Solve the following equations for $0 \le x \le 2\pi$.

   (a) $3\cos^2 x - \cos x = 0$

   (b) $2\sin^2 x + \sin x - 1 = 0$

   (c) $4\cos^3 x - 4\cos^2 x - 3\cos x + 3 = 0$

   (d) $\tan^2 x + 2\tan x + 1 = 0$

Section 3 Trigonometric identities

This section states and proves some common trig identities.

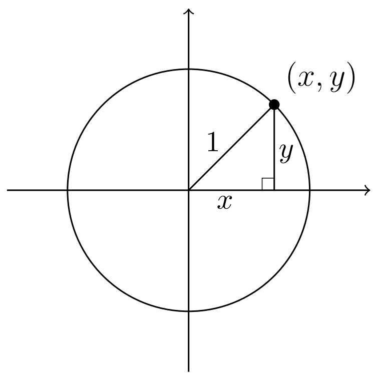
Pythagorean Identities

① $\cos^2\theta + \sin^2\theta = 1$

② $1 + \tan^2\theta = \sec^2\theta$

③ $1 + \cot^2\theta = \csc^2\theta$

Proof of ①: Consider a circle of radius 1 centred at the origin.



- Let θ be the angle measured anticlockwise for the positive x-axis.
- Using trig ratios we see that $x = \cos\theta$ and $y = \sin\theta$.
- By Pythagoras' Theorem $x^2 + y^2 = 1$. i.e. $\cos^2\theta + \sin^2\theta = 1$. □

Proof of ②: Divide both sides of identity ① by $\cos^2\theta$ and the result follows. □

Proof of ③: Divide both sides of identity ① by $\sin^2\theta$ and the result follows. □

Sum and Difference Identities

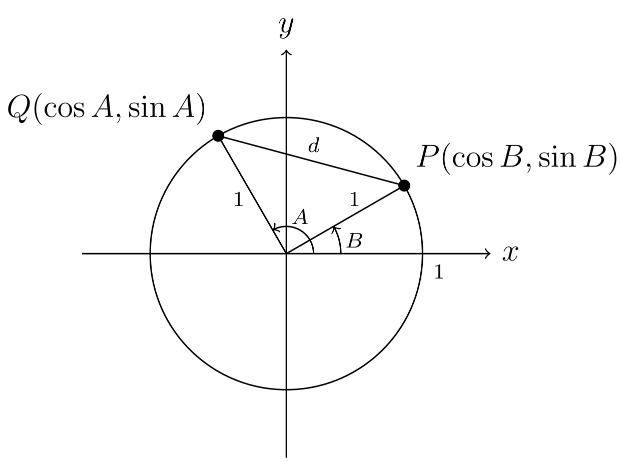
④ $\sin(A+B) = \sin A\cos B + \cos A\sin B$

⑤ $\sin(A-B) = \sin A\cos B - \cos A\sin B$

⑥ $\cos(A+B) = \cos A\cos B - \sin A\sin B$

⑦ $\cos(A-B) = \cos A\cos B + \sin A\sin B$

Proof of ④ – ⑦: We first prove ⑦. Consider the following circle of radius 1 with angles A and B as shown.



Note that we can label point P as $(\cos B, \sin B)$ and point Q as $(\cos A, \sin A)$ by using trig ratios. We can calculate the distance d using two methods.

Using the distance formula we see that

$$\begin{aligned} d^2 &= (\cos B - \cos A)^2 + (\sin B - \sin A)^2 \\ &= \cos^2 B - 2\cos A\cos B + \cos^2 A + \sin^2 B - 2\sin A\sin B + \sin^2 A \\ &= (\cos^2 B + \sin^2 B) + (\cos^2 A + \sin^2 A) - 2(\cos A\cos B + \sin A\sin B) \\ &= 2 - 2(\cos A\cos B + \sin A\sin B) \end{aligned}$$

Using the cosine rule we have

$$\begin{aligned} d^2 &= 1^2 + 1^2 - 2(1)(1)\cos(A-B) \\ &= 2 - 2\cos(A-B) \end{aligned}$$

Equating these we see that

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

thus proving sum and difference identity ⑦.

We can use identity ⑦ to deduce the remaining identities. We have

$$\begin{aligned} \cos(A+B) &= \cos(A-(-B)) \\ &= \cos A\cos(-B) + \sin A\sin(-B) \\ &= \cos A\cos B - \sin A\sin B \end{aligned}$$

$$\begin{aligned} \sin(A+B) &= \cos\big(\tfrac{\pi}{2} - (A+B)\big) \\ &= \cos\big(\tfrac{\pi}{2} - A - B\big) \\ &= \cos\big(\tfrac{\pi}{2} - A\big)\cos B + \sin\big(\tfrac{\pi}{2} - A\big)\sin B \\ &= \sin A\cos B + \cos\big(\tfrac{\pi}{2} - \big(\tfrac{\pi}{2} - A\big)\big)\sin B \\ &= \sin A\cos B + \cos A\cos B \end{aligned}$$

$$\begin{aligned} \sin(A-B) &= \sin(A+(-B)) \\ &= \sin A\cos(-B) + \cos A\sin(-B) \\ &= \sin A\cos B - \cos A\sin B \end{aligned}$$

We have now established identities ④ − ⑥. □

Double Angle Identities

⑧ $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$= 2\cos^2\theta - 1$

⑨ $\sin 2\theta = 2\sin\theta\cos\theta$

Proof of ⑧ and ⑨: using the sum and difference identities we can prove the double angle identities. For instance,

$$\begin{aligned}\cos 2\theta &= \cos(\theta+\theta)\\ &\cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Replacing $\cos^2\theta$ by $1-\sin^2\theta$ (Pythagorean identity ①), we can see that $\cos 2\theta = 1 - 2\sin^2\theta$.

Replacing $\sin^2\theta$ by $1-\cos^2\theta$ (Pythagorean identity ①), we can see that $\cos 2\theta = 2\cos^2\theta - 1$.

We also have

$$\begin{aligned}\sin 2\theta &= \sin(\theta+\theta)\\ &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ &= 2\sin\theta\cos\theta.\end{aligned}$$

We have now established identities ⑧ and ⑨. □

Half Angle Identities

⑩ $\cos^2\left(\dfrac{\theta}{2}\right) = \dfrac{1+\cos\theta}{2}$

⑪ $\sin^2\left(\dfrac{\theta}{2}\right) = \dfrac{1-\cos\theta}{2}$

To prove the half angle identities we begin by rearranging the double angle identities.

Proof of ⑩: We take the double angle identity $\cos 2\theta = 2\cos^2\theta - 1$ to obtain

$$\begin{aligned}2\cos^2\theta &= \cos 2\theta + 1\\ \text{i.e.}\quad \cos^2\theta &= \frac{\cos 2\theta + 1}{2}\\ \text{i.e.}\quad \cos^2\left(\frac{\theta}{2}\right) &= \frac{\cos\theta + 1}{2}\end{aligned}$$

Proof of ⑪: We take the double angle identity $\cos 2\theta = 1 - \sin^2\theta$ to obtain

$$
\begin{aligned}
2\sin^2\theta &= 1 - \cos 2\theta \\
\text{i.e.} \quad \sin^2\theta &= \frac{1-\cos 2\theta}{2} \\
\text{i.e.} \quad \sin^2\left(\frac{\theta}{2}\right) &= \frac{1-\cos\theta}{2}
\end{aligned}
$$

We have now established identities ⑩ and ⑪. □

Example 1 : Simplify $2\cos x\cos 2x\sin 3x - 2\sin x\sin 2x\sin 3x$.

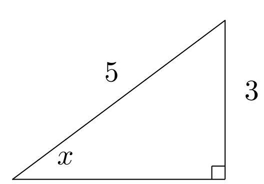
$$
\begin{aligned}
&2\cos x\cos 2x\sin 3x - 2\sin x\sin 2x\sin 3x \\
&\quad = 2\sin 3x(\cos x\cos 2x - \sin x\sin 2x) && \text{(factorise)} \\
&\quad = 2\sin 3x(\cos(x+2x)) && \text{(difference identity ⑥)} \\
&\quad = 2\sin 3x\cos 3x \\
&\quad = \sin 2(3x) && \text{(double angle identity ⑨)} \\
&\quad = \sin 6x
\end{aligned}
$$

Example 2 : Show that $2\cos^2 2x - \cos^2 x - \sin^2 x = \cos 4x$.

$$
\begin{aligned}
2\cos^2 2x - \cos^2 x - \sin^2 x &= 2\cos^2 2x - (\cos^2 x + \sin^2 x) \\
&= 2\cos^2 2x - 1 && \text{(Pythagorean identity ①)} \\
&= \cos 2(2x) && \text{(double angle identity ⑧)} \\
&= \cos 4x
\end{aligned}
$$

Example 3 : Given $\sin x = \frac{3}{5}$ for $\frac{\pi}{2} \leq x \leq \pi$, find: (i) $\cos x$ and (ii) $\sin 2x$.

Since $\frac{\pi}{2} \leq x \leq \pi$, we know x lies in the second quadrant. Also, since $\sin x = \frac{3}{5}$, we can form the following right angled triangle.



Using Pythagoras we see that the horizontal side is 4. We must have $\cos x = -\frac{4}{5}$, since cosine is negative in the second quadrant. So

$$
\begin{aligned}
\sin 2x &= 2\sin x\cos x \\
&= 2(\tfrac{3}{5})(-\tfrac{4}{5}) \\
&= -\tfrac{4}{25}
\end{aligned}
$$

Example 4 : Use the appropriate half angle identity to find the exact value of $\sin(\frac{\pi}{8})$.

The half angle identity for sine is ⑪ is

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$
$$\text{i.e.} \quad \sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Now,

$$\begin{aligned}
\sin\left(\frac{\pi}{8}\right) &= \sin\left(\frac{\pi/4}{2}\right) \\
&= \pm\sqrt{\frac{1-\cos(\pi/4)}{2}} \\
&= \pm\sqrt{\frac{1-1/\sqrt{2}}{2}} \\
&= \pm\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \\
&= \pm\sqrt{\frac{2-\sqrt{2}}{4}} \\
&= \pm\frac{\sqrt{2-\sqrt{2}}}{2}
\end{aligned}$$

Since $\frac{\pi}{8}$ lies in the first quadrant, where sine is positive, we must have

$$\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Exercises:

1. Simplify the following

   (a) $\cos 5x \sin x - \cos x \sin 5x$

   (b) $\dfrac{\sin^2 x + \cos^2 x + \tan^2 x}{\sec^2 x}$

   (c) $3 - 6\sin^2(x/8)$

   (d) $\dfrac{3\cot 2x \sin x \cos x}{\cos^2 x - \sin^2 x}$

   (e) $2\sin x \cos x - 4\sin^3 x \cos x$

   (f) $\frac{1}{2}\big(\cos(x/2) + 2\sin^2(x/4) - 1\big)$

2. Use the addition formulas to find the exact value of the following.

   (a) $\cos\left(\dfrac{7\pi}{12}\right)$ (b) $\sin\left(\dfrac{14\pi}{12}\right)$ (c) $\tan\left(\frac{7\pi}{6}\right)$

3. Use the half angle identities to calculate the exact value of the following.

   (a) $\cos\left(\frac{\pi}{8}\right)$ (b) $\cos\left(\frac{\pi}{12}\right)$ (c) $\sin\left(-\frac{\pi}{12}\right)$

4. Given $\tan x = \frac{5}{12}$ for $0 \leq x \leq \frac{\pi}{2}$, evaluate the following.

   (a) $\sin x$ (b) $\cos x$ (c) $\sin 2x$ (d) $\cos 2x$ (e) $\cos 3x$

Exercises for Worksheet 4.6

1. Solve the following equations.

   (a) $\tan x = -\sqrt{3}$, $-\pi \le x \le \pi$

   (b) $\sin 2x = \frac{1}{2}$, $0 \le x \le 2\pi$

   (c) $4\sin^3 x - \sin x = 0$, $0 \le x \le 2\pi$

   (d) $\sec^2 x - 3\sec x + 2 = 0$, $0 \le x \le 2\pi$

2. Sketch the following curves.

   (a) $y = -2\cos\left(\dfrac{x}{3}\right)$

   (b) $y = 1 + \sin\left(x + \dfrac{\pi}{3}\right)$

   (c) $y = \tan 3\left(x - \dfrac{\pi}{6}\right)$

3. Prove the following.

   (a) $\cos 2x \sin x - \cos x \sin 2x = -\sin x$

   (b) $\dfrac{2\tan x - 2\sin^2 x \tan x}{\sin 2x} = 1$

   (c) $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \sin x}{\cos x}$

4. Suppose α lies in the third quadrant, β lies in the fourth quadrant and $\sin\alpha = -\frac{4}{5}$ with $\cos\beta = \frac{12}{15}$. Find the following.

   (a) $\sin(\alpha + \beta)$

   (b) $\cos(\alpha + \beta)$

   (c) $\tan(\alpha + \beta)$

5. Suppose $\cos x = \frac{2}{3}$ where $\frac{3\pi}{2} \le x \le 2\pi$. Find the exact value of the following.

   (a) $\cos 3x$

   (b) $\sin 3x$

6. Use the half angle identity to find the exact value of $\sin(-\frac{3\pi}{8})$.

7. Use the appropriate identities to find the exact values of the following.

   (a) $\cos\left(\dfrac{1}{2}\left(\dfrac{\pi}{4} - \dfrac{\pi}{3}\right)\right)$

   (b) $\sin\left(2\left(\dfrac{\pi}{6} + \dfrac{3\pi}{4}\right)\right)$

Worksheet 4.7 Modifying Functions - Families of Graphs

Section 1 Domain, range and functions

We first met functions in Sections 3.1 and 3.2. We will now look at functions in more depth and discuss their domain and range more formally.

Defining Functions

A function f is specified by a rule and two sets. These sets are the *domain*, previously discussed in worksheet 3.2, which we'll call A, and the *codomain* which we'll call B. The domain of a function contains all the values that we can input into the function. The codomain is a set containing all possible values the function could achieve. The codomain is usually given.

We can think of our function as a mapping from the domain to the codomain and we usually write this as

$$f : A \to B$$

where

$$\begin{aligned} A &= \text{domain} \\ B &= \text{codomain.} \end{aligned}$$

Example 1 : Consider $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$.

We say that f takes real numbers (from the domain), applies the rule and maps them to other real numbers in the codomain. Here the rule says that we square what we put into our function.

So for example $f(2) = 4$. That is, f maps 2 from the domain to 4 in the codomain. Another example is $f(-1) = 1$. In this case f maps -1 in the domain to 1 in the codomain.

Example 2 : Consider $f : [-1, 3) \to [1, 10)$, where $f(x) = x^2 + 1$.

The above function f has domain $[-1, 3)$ and codomain $[1, 10)$.
The rule of this function is that we take some element x of the domain and then evaluate $x^2 + 1$ to find $f(x)$. For example $f(\frac{1}{2}) = 1\frac{1}{4}$.

The last set we will need to define is the *range* of a function. The range, also talked about in worksheet 3.2, is a subset of the codomain, and contains all of the values that f actually attains. The range is always contained in the codomain, but the codomain might not necessarily be in the range (it can contain more things). In Example 2 the range of f is the same as the codomain, $[1, 10)$.

Example 3 : Consider again $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$.

Since we square all our inputs, we will never get a negative number as an output. In other words, $f(x)$ will only attain 0 and positive values. So the range of $f(x)$ is $[0, \infty)$. Notice this is a subset of the given codomain $\mathbb{R}$.

Example 4 : Consider $f : [0, 3) \to \mathbb{R}$, where $f(x) = x^3 - 7$. Here the domain is $[0, 3)$, the range is $[-7, 20)$ and the codomain is $\mathbb{R}$ (given). Note that the range is a subset of the codomain.

All functions have only one value of $f(x)$ for each value of x.

Example 5 : Find the domain of $f(x) = \frac{1}{x-5}$.
We can never divide by 0, thus for x to be in the domain it must satisfy $x - 5 \neq 0$. So the domain of the function is $x \neq 5$.

Set Notation

Another way of writing the domain from Example 5 is to use set notation. We say

$$\text{Domain} = \big\{x \in \mathbb{R} : x \neq 5\big\},$$

which we read as 'the set of all real numbers x, such that x is not equal to 5'. We use the brackets $\{\dots\}$ to respresent our set and the symbol $\in$ to mean 'belongs to'. The colon ' : ' may be read as either 'such that' or 'with the property that'.

Example 6 : Find the domain and range of $f(x) = \sqrt{x+3}$.
We can only take the square root of positive numbers or 0, so to be in the domain x must satisfy $x + 3 \geq 0$, which gives domain $[-3, \infty)$. The square root function always gives a positive or 0 result, so $\sqrt{x+3} \geq 0$ so the range of the function is $[0, \infty)$.

We can write this using set notation as

$$\text{Domain} = \big\{x \in \mathbb{R} : x \geq -3\big\}$$
$$\text{Range} = \big\{y \in \mathbb{R} : y \geq 0\big\}.$$

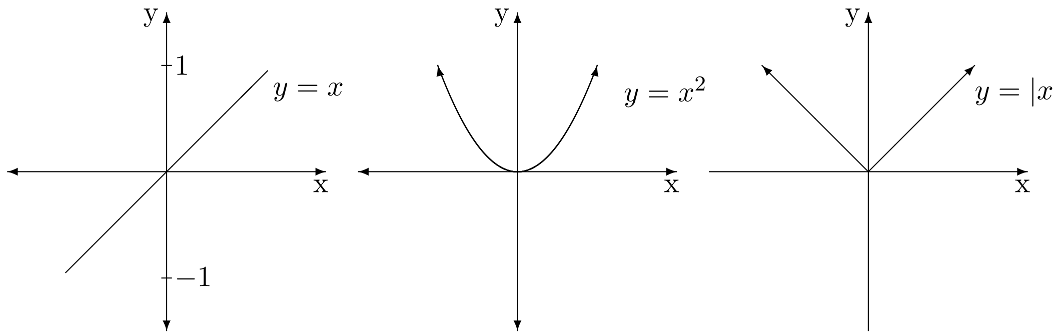
Exercises:

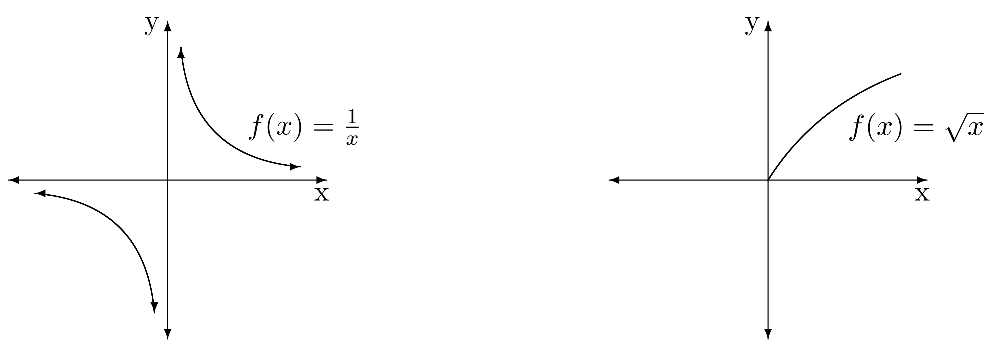
1. State the domain and codomain and find the range of the following functions
   (a) $f : (0, 5) \to \mathbb{R}, f(x) = x^2 - 2$.
   (b) $g : [-1, 7] \to \mathbb{R}, g(x) = x^2 - 2x + 1$.
2. Find a suitable domain and the related codomain for $f(x) = \sqrt{x - 1}$.
3. Find the domain of the functions below and express it in set notation.
   (a) $f(x) = \sqrt{x}$
   (b) $f(x) = \frac{1}{\sqrt{x}}$
   (c) $f(x) = \frac{1}{3x-2}$
   (d) $f(x) = x^3 - 1$
   (e) $f(x) = \sqrt{x - 7}$
   (f) $f(x) = \sqrt{x^2 - 16}$
4. Find the domain and range of
   (a) $f(x) = x^2 + 1$
   (b) $f(x) = \sqrt{x + 9}$
   (c) $f(x) = 4x - 2$

Section 2 Modifying Functions – Translations

You may have noticed that it is often easier to find the range of a function by looking at its graph. In fact the graph of a function can give us the big picture on how the function behaves. The next two sections look at modifying known functions to quickly and easily sketch other graphs.

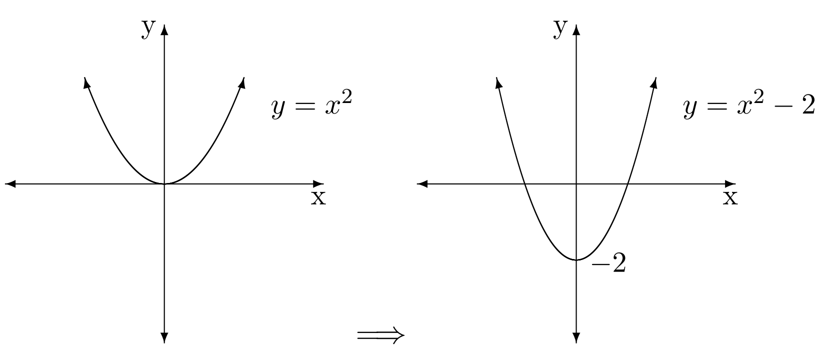
We know how to draw the standard graphs of some basic functions, for example





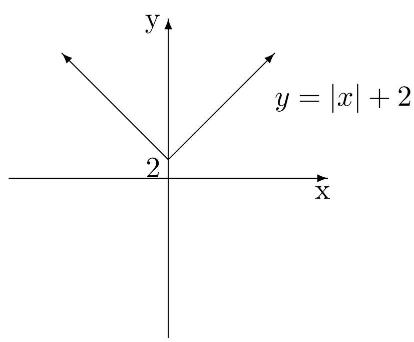
This worksheet will show you how to easily and quickly draw modified versions of these graphs.

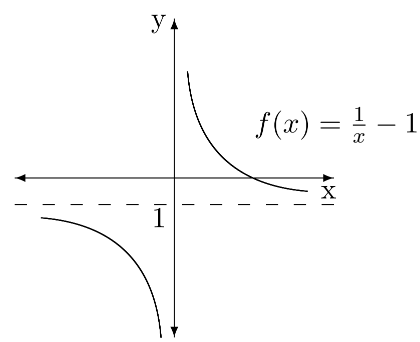
The first kind of modification is one that occurs on the y-coordinate. For example we might want to sketch $y = x^2 - 2$. This graph is taken by picking some value of x, squaring it and then subtracting 2. This is the same for every value of x, so this is just the graph $y = x^2$ shifted down by 2 units.



Definition 1 : The modification $y = f(x) + a$ is drawn by shifting the graph of $y = f(x)$ up by a units. Similarly the modification $y = f(x) - b$ is drawn by shifting the graph down b units.

Example 1 : Sketch the graphs $y = |x| + 2$ and $y = \frac{1}{x} - 1$. The first graph is a shift up of 2 units, and the second graph is a shift down of one unit.





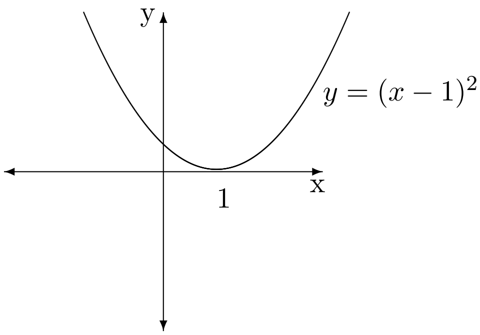
The other kind of modification is one that occurs on the x coordinate. For example suppose we want to sketch

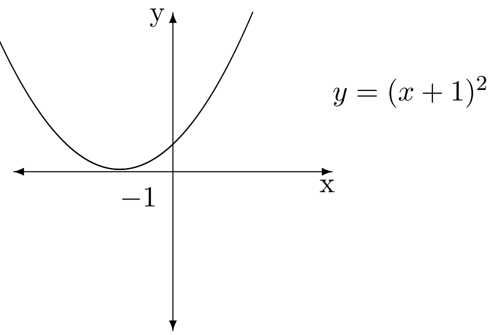
(a) $y = (x-1)^2$

(b) $y = \frac{1}{x+2}$

(c) $y = \sqrt{x-3}$

The key thing to notice here is that the order of operations tells us that in (a) we subtract 1 from x and then square the expression. Similarly in (b) we add 2 to x and then apply the function. Finally in (c) we subtract 3 from x and then take the square root. All of these amount to essentially the same thing, i.e. we are adding or subtracting a number before applying the basic function. To see this, think for a moment about how the graphs are drawn. Let's look at (a) - going from the original function $y = x^2$ to the new function $y = (x-1)^2$. In the original function we take some particular value for x, let's say $x = 4$, and **then** square it to get the y value. So we have a point on the original graph $(4, 16)$. In comparison, looking at our new function and taking the same x value, we subtract 1 from it **before** squaring so the y value is $y = 3^2 = 9$ and the point on our new graph is $(4, 9)$. So the y value is the same as the y value 1 unit to the left in the non-modified graph.

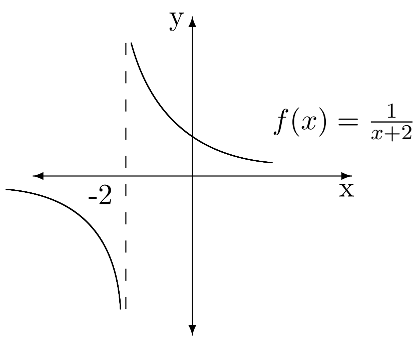


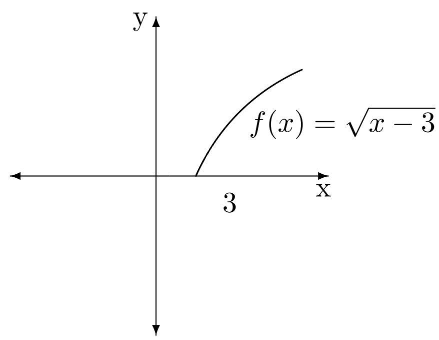


A simple way to state a rule would be to say:

Definition 2 : For a modification to a function on the x coordinate of $f(x+a)$ the graph is drawn by shifting $f(x)$ to the left by a units. Similarly the graph $f(x-b)$ is drawn by shifting $f(x)$ to the right by b units.

The other graphs look like:





The modifications we have looked at in this section have either shifted our original function vertically or horizontally. We call this kind of modification a translation.

Exercises:

1. On the same diagram sketch the graphs of $y = x^2$ and $y = x^2 + 3$.
2. Sketch
   (a) $y = \sqrt{x+5}$
   (b) $y = \sqrt{x} + 5$
   (c) $y = |x+20|$
   (d) $y = |x| + 20$
   (e) $y = \frac{1}{x-7}$
   (f) $y = 3 + \frac{1}{x-7}$

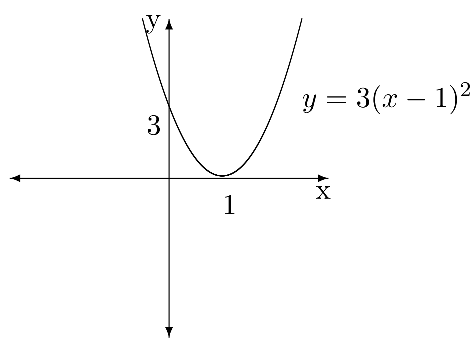
Section 3 Other Modifications

There are three more standard modifications to consider, the first is multiplying the function by a constant. This modification takes the original y values of the functions and changes them by the scalar that we are multiplying by.

Example 1 :

$$y = 3(x-1)^2$$

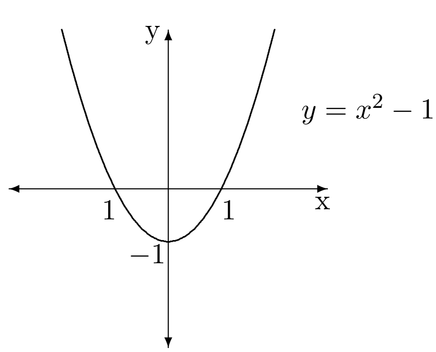
In this example the number 3 is the scalar of multiplication.

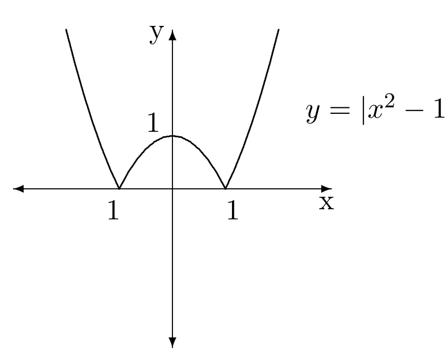


The only difference between this example and the previous drawing of $y = (x-1)^2$ is that this function is steeper

The next modification we'll talk about is taking absolute values. This modification takes any positive y values and leaves them unchanged, and takes any negative values making them positive with the same value. This is the same as putting a mirror along the x axis, and drawing any values below the axis at their mirrored position above the x axis.

Example 2 : Sketch the graph of $y = |x^2 - 1|$.





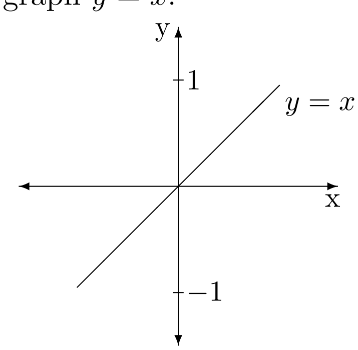
The last modification we will look at is taking reciprocals. The reciprocal of x is $\frac{1}{x}$, we just flip the fractions over (thinking of x as $\frac{x}{1}$). Here are some numbers and their reciprocals.

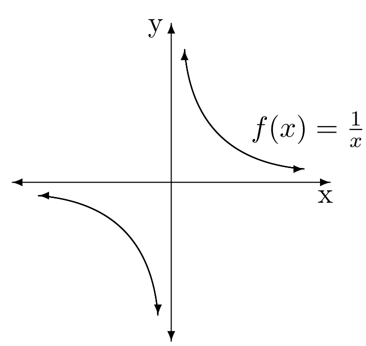
Number (x)	Reciprocal $\frac{1}{x}$	Number (x)	Reciprocal $\frac{1}{x}$
1	1	$\frac{1}{4}$	4
2	$\frac{1}{2}$	$\frac{1}{2}$	2
3	$\frac{1}{3}$	$\frac{1}{100}$	100
10	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{5}{2}$
100	$\frac{1}{100}$	$\frac{2}{55}$	$\frac{55}{2}$

Note that the smaller the number, the larger the reciprocal and the larger the number the smaller the reciprocal.

Note that 0 has no reciprocal, but as x gets closer to 0 its reciprocal gets larger (approaches ∞). If it gets close to zero and is negative then its reciprocal will approach $-\infty$, if it is positive then it approaches ∞. Similarly as x gets large (approaches ∞) the reciprocal approaches 0.

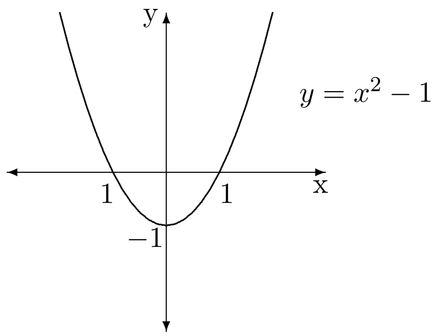
Example 3 : We can see how to draw the reciprocal function $y = \frac{1}{x}$ by using the graph $y = x$.

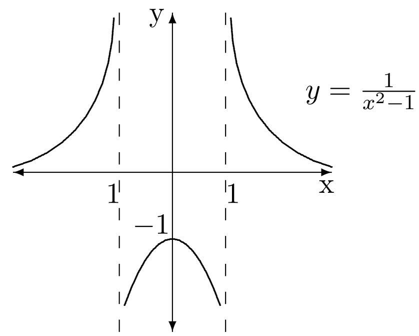




On the graph of $y = \frac{1}{x}$ notice there is a break along the line $x = 0$. This is because x cannot take the value of 0, however as x gets closer to 0, $\frac{1}{x}$ approaches $\pm\infty$ respectively. We call the line $x = 0$ a vertical asymptote.

Example 4 : Sketch the graph of $y = \frac{1}{x^2-1}$.





Exercises:

1. Sketch

   (a) $y = 3(x-2)^2 + 1$

   (b) $y = 7 - \frac{4}{x+3}$

   (c) $y = 2|x-7| + 10$

   (d) $y = \sqrt{3x-2} - 1$

2. Sketch the graph of $y = x^2$. Using this and considering reciprocals sketch the graph of $y = \frac{1}{x^2}$.

3. Let $f(x) = 2 - x^2$. Sketch

   (a) $y = f(x) + 3$

   (b) $y = |f(x)|$

   (c) $y = f(x+3)$

   (d) $y = 1 - f(x)$

4. Sketch $f(x) = \sqrt{x-3} + 2$ and find its domain and range.

Exercises for Worksheet 4.7

1. Sketch the following functions and find their domains and ranges.

   (a) $y = \frac{1}{x-1}$
   (b) $y = |x-7| - 3$
   (c) $y = \sqrt{3x-14}$
   (d) $y = x^2 + 6x - 1$ (Hint: complete the square)

2. Sketch the following functions.

   (a) $y = x^2 + 3$
   (b) $y = (x+3)^2$
   (c) $y = 1 + (x+3)^2$
   (d) $y = 1 - (x+3)^2$
   (e) $y = \frac{1}{(x+3)^2}$
   (f) $y = 5(x+3)^2 - 1$

3. Sketch the following functions.

   (a) $y = 3 + \sqrt{4x-9}$
   (b) $y = \left|\frac{1}{x^2-4}\right|$
   (c) $y = \frac{x-1}{x+2}$
   (d) $y = \left|\frac{3x-1}{3x-4}\right|$

4. In this question $f(x) = \frac{1}{x-2}$. Sketch the following.

   (a) $y = f(x)$
   (b) $y = f(x+3)$
   (c) $y = f(x) + 3$
   (d) $y = |f(x) + 3|$
   (e) $y = \frac{1}{f(x)}$

Worksheet 4.8 Composite and Inverse Functions

Section 1 Composition

We'll begin by defining the composition function $f \circ g(x) = f(g(x))$, which is read as "f of g of x". Another helpful way to think about these is to call them "a function (f) of a function (g)". To calculate this function for a given x, first evaluate $g(x)$, which will give us a number, and then take this number and apply the function f to it giving $f(g(x))$. Note: For $f \circ g(x)$ to exist x must be in the domain of g and $g(x)$ must be in the domain of f. This is expanded on in Section 3 of Chapter 3.1.

Example 1 : Find the composition of the two functions $f(x) = x^2 - 2x + 3$ and $g(x) = 2x - 1$.

$$(f \circ g)(x) = (2x-1)^2 - 2(2x-1) + 3 = 4x^2 - 8x + 6$$

$$(g \circ f)(x) = 2(x^2 - 2x + 3) - 1 = 2x^2 - 4x + 5.$$

Notice that $f \circ g \neq g \circ f$.

Example 2 : Consider the functions $p(x) = \sqrt{x-1}$ and $q(x) = x^2$. Find $(q \circ p)(x)$.

Firstly notice $p(x)$ has domain $[1, \infty)$ and range $[0, \infty)$ and $q(x)$ has domain $\mathbb{R}$ and range $[0, \infty)$.

Look at $(q \circ p)(x) = q(p(x)) = (\sqrt{x-1})^2 = x - 1$. When finding the domain of $(q \circ p)(x)$ we need to consider what is happening in this composite. Here the function $p(x)$ is applied first, followed by $q(x)$.

So we start by inputting the domain $[1, \infty)$ of $p(x)$ such that its range $[0, \infty)$ becomes the input of $q(x)$. Then $q(x)$ squares these values to give the overall output of the composite. Stringing these observations together tells us that the domain of $(q \circ p)(x)$ is $[1, \infty)$ and its corresponding range is $[0, \infty)$.

Notice that we are incorrectly tempted to look at $(q \circ p)(x) = x - 1$ and claim its domain is $\mathbb{R}$ and its range is $\mathbb{R}$. However as this is a composite of two functions its input and output values depend on the domain and range of the two individual functions $q(x)$ and $p(x)$.

Sometimes it is necessary to restrict the domain of g, so that $f \circ g$ will exist.

Exercises:

1. Consider $f(x) = 4 - x^2$, $g(x) = \sqrt{x+3}$, $h(x) = \frac{1}{2x}$. Evaluate the following.
   (a) $(f \circ g)(1)$
   (b) $(g \circ h)(1)$
   (c) $(f \circ g)(x)$
   (d) $(g \circ h)(x)$
   (e) $(h \circ g)(x)$
   (f) $(f \circ g)(x^2)$
   (g) $(f \circ g \circ h)(x)$
2. Using the functions given in the previous exercise, explain why $(f \circ g)(-4)$ does not exist.
3. Let $s(x) = \sqrt{x}$ and $t(x) = x^2 + 2x + 1$. Evaluate $(s \circ t)(x)$ and state its domain and range.
4. $f(x) = \sqrt{\frac{1}{x^2+2}}$. Write $f(x)$ as the composition of two or more functions.

Section 2 Inverse Functions

Let us introduce the concept of inverse functions by looking at some examples.

Example 1 : $f(x) = x + 2, \quad g(x) = x - 2$
$f(x)$ adds 2 to everything we put into it.
$g(x)$ subtracts 2 from everything we put into it.
What happens when we take $f \circ g$?
$f(g(x)) = f(x-2) = x - 2 + 2 = x$. $(f \circ g)(x)$ takes the input x, first subtracts 2 then adds 2 so we are back to what we started with.
Also $(g \circ f)(x) = g(f(x)) = g(x+2) = x + 2 - 2 = x$. So $f(x)$ and $g(x)$ "undo" each other.

Example 2 : $h(x) = 3x, \quad k(x) = \frac{x}{3}$
Here $h(x)$ multiplies what we put in by 3 and $k(x)$ divides what we put in by 3.
$(h \circ k)(x) = h\left(\frac{x}{3}\right) = 3\frac{x}{3} = x$ and $(k \circ h)(x) = k(3x) = \frac{(3x)}{3} = x$.
So $(h \circ k)(x)$ takes the input x, divides by 3, then multiplies by 3, thus returning back to x. $(k \circ h)(x)$ takes the input x, multiplies by 3, then divides by 3, thus returning back to x. So $h(x)$ and $k(x)$ "undo" each other.

Notice in each of examples 1 and 2 the operations performed are "opposites" so the functions "undo" each other. Similarly we have example 3.

> Example 3 : $p(x) = x^3, \quad q(x) = x^{1/3}$
> $p(x)$ cubes what we put in and $q(x)$ takes the cube root of what we put in. So $p(x)$ and $q(x)$ will undo each other and $(p \circ q)(x) = x = (q \circ p)(x)$.

We can see that certain pairs of functions have the important property that they undo each other. We call functions that undo each other *inverses*. So $p(x)$ and $q(x)$ are inverses since $(p \circ q)(x) = x$ and $(q \circ p)(x) = x$.

To represent this special relationship they have with each other we rename them. For $p(x) = x^3$ we say

$$p^{-1}(x) = x^{\frac{1}{3}}.$$

Here we use a superscript of -1 to represent that we are talking about the inverse of p. [NOTE: $p^{-1}(x) \neq \frac{1}{p(x)}$] So we say that $(p \circ p^{-1})(x) = (p^{-1} \circ p)(x) = x$.

> **Definition 1** : The inverse function for f is the unique function f^{-1} that satisfies $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

Looking back, from example 1 we have $f(x) = x + 2$, $f^{-1}(x) = x - 2$ and $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. While from example 2 we have example 2) $h(x) = 3x$, $h^{-1}(x) = \frac{x}{3}$ and $(h \circ h^{-1})(x) = (h^{-1} \circ h)(x) = x$.

Exercises:

1. Show the following function pairs are inverses:
   (a) $f(x) = x^2$ and $g(x) = \sqrt{x}$
   (b) $f(x) = \frac{1}{x^2-1}$ and $g(x) = \sqrt{\frac{1}{x} + 1}$
   (c) $f(x) = 2x - 2$ and $g(x) = \frac{x}{2} + 1$
2. Which pairs of the following functions are inverses?
   (a) $y = x^2 - 2$
   (b) $y = x - 7$
   (c) $y = x^2 - 4x + 4$

(d) $y = \sqrt{x} + 2$
(e) $y = \sqrt{x + 2}$
(f) $y = x + 7$

Section 3 Finding Inverses

There is a particular technique we can use to find inverses of functions. Let's look at an example to see how this works.

Example 1 :

$$f(x) = \frac{2x-3}{7}.$$

To find the inverse of $f(x)$, let's rewrite it so that we call the output values y, i.e. $y = \frac{2x-3}{7}$. Since the inverse function will undo the original, we expect the outputs of the inverse to bring us back to the inputs of the original, and vice versa. So for our inverse function we expect

$$x = \frac{2y-3}{7}$$

i.e. we swap the x and y values which represent the inputs and outputs. To find the inverse function we now make y the subject.

$$\begin{aligned} x &= \frac{2y-3}{7} \\ 7x &= 2y - 3 \\ 7x + 3 &= 2y \\ y &= \frac{7x+3}{2}, \end{aligned}$$

so our inverse function is $y = \frac{7x+3}{2}$. We write $f^{-1}(x) = \frac{7x+3}{2}$.

Notice we have a simple technique to find inverses - we swap x and y in the original function and then rearrange to make y the subject.

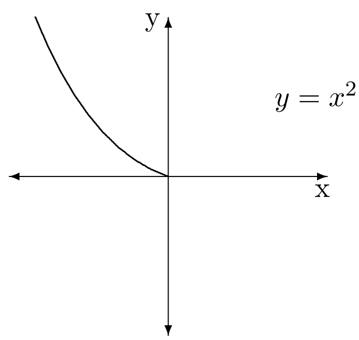
Example 2 : $f(x) = 3x - 11$
Let $y = 3x-11$. To find the inverse we write $x = 3y-11$ and rearrange $x+11 = 3y$ giving $y = \frac{x+11}{3}$. So

$$f^{-1}(x) = \tfrac{x+11}{3}.$$

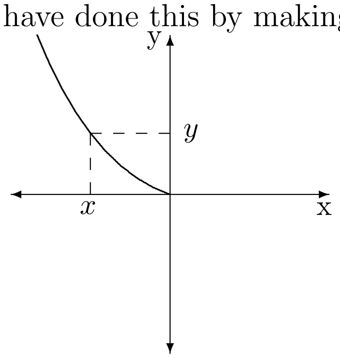
Suppose $f : A \to B$ where A is the domain and B is the range. Then $f^{-1} : B \to A$. In fact we have to take particular note of the domain and range when finding inverse functions especially in certain situations.

Consider $f(x) = x^2$ where $f : \mathbb{R} \to [0, \infty)$. To find the inverse we write $y = x^2$ then swap $x = y^2$ then rearrange $y^2 = x$ to get $y = \pm\sqrt{x}$. Here $f^{-1}(x) = +\sqrt{x}$ and $f^{-1}(x) = -\sqrt{x}$. We cannot have 2 different inverse functions for $f(x) = x^2$. If $f^{-1}(x)$ is both $\pm\sqrt{x}$ then it is not a function at all. So in this situation we need to make sure that $f(x)$ is defined correctly in the first place so that the inverse can be taken. Since we don't want two different inverses, we need to restrict the domain of $f(x)$.

Let's take $f : (-\infty, 0] \to [0, \infty), f(x) = x^2$



Therefore $f^{-1} : [0, \infty) \to (-\infty, 0]$. By restricting the domain we have removed of the problem of getting more than 1 inverse. We have done this by making sure that for each y-value there is only one corresponding x-value.

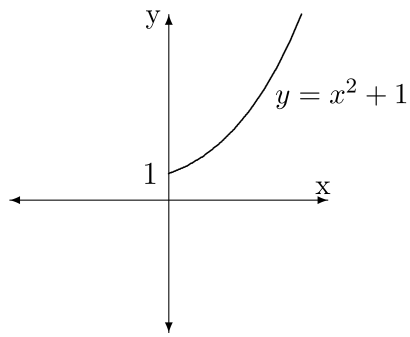


Now by looking at the range of f^{-1} (the negative reals) we can therefore see that we need $f^{-1}(x) = -\sqrt{x}$ here.

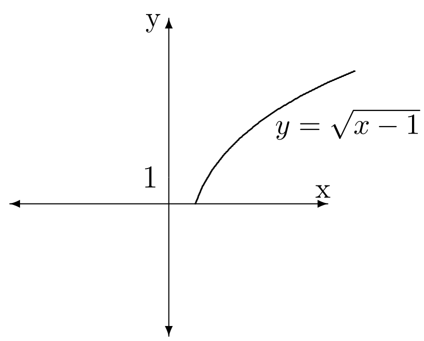
There are times that we need to restrict the domain and codomain of a function so that is is possible to find its inverse. In the previous example we have done this by making sure f is one-to-one i.e. made sure that for each output there is only one input corresponding to it. We also need our function to be onto i.e. the function gives outputs across the whole domain, meaning that the codomain = range. (We didn't have to worry about this in the previous example as the codomain = range was given). By restricting our function so that it is both 1-1 and onto we will be able to find an inverse.

Example 3 : Find the inverse of the function $f(x) = x^2 + 1$ with domain $[0, \infty)$ (so that the range of f is $[1, \infty)$).

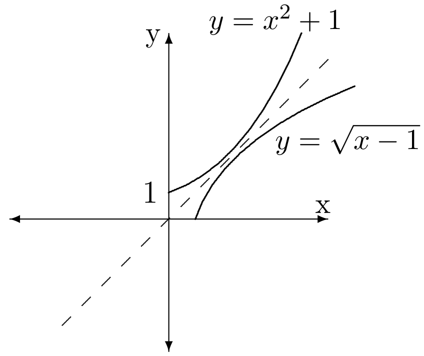
From a quick sketch we see that f is 1-1 and onto.



We need $f : [0, \infty) \to [1, \infty)$ so $f^{-1} : [1, \infty) \to [0, \infty)$. We find the inverse as before, start by swapping variables to $x = y^2 + 1$ and then rearrange to get $y = \pm\sqrt{x-1}$. By looking at the range of f^{-1}, the positive reals, we see that we need to take $y = +\sqrt{x-1}$ as the inverse i.e. $f^{-1}(x) = \sqrt{x-1}$.



We can observe that a special property of inverse functions is that they are symmetrical about the line $y = x$



Exercises:

1. Find the inverses of these functions and sketch each function and its inverse on the same graph.

   (a) $y = 3x$
   (b) $y = x - 7$
   (c) $y = \frac{x}{2} + 17$
   (d) $y = \frac{1-5x}{4}$
   (e) $y = \frac{3}{x+8}$
   (f) $y = x^3$

2. Find the inverses of these functions and state their domain and range.

   (a) $y = \sqrt{x} + 2$
   (b) $y = \sqrt{x+2}$

3. By suitably restricting the domain, find the inverse of $y = x^2 - 2$ and state its domain and range.

Exercises for Worksheet 4.8

1. For the following pairs of functions f and g find the composition functions $f \circ g$ and $g \circ f$. Also find the domains of the two composition functions.

   (a) $f = 2x - 5$ and $g = x^2 - 3x$

   (b) $f = \sqrt{3x - 1}$ and $g = x^2$

   (c) $f = \sqrt{3x - 1}$ and $g = \frac{1}{x}$

   (d) $f = \sqrt{3x - 1}$ and $g = \frac{x^2+1}{x}$

   (e) $f = x^2 - 3$ and $g = |x|$

2. Write each of the following functions as the composition of two or more simpler functions.

   (a) $\sqrt{x^3 - 1}$

   (b) $(3x - 4)^3$

   (c) $\frac{1}{x^2-1}$

   (d) x

3. Find the inverses of the following functions.

   (a) $y = 3x + 2$

   (b) $y = \frac{1}{4-x}$

   (c) $y = \frac{x+2}{x+5}$

   (d) $y = x^3 + 1$

4. Find the inverses of the following functions and specify the domain of these inverses.

   (a) $f(x) = x^2 - 2x + 3$, domain $[1, \infty)$

   (b) $h(x) = |x|$, domain $(-\infty, 0)$

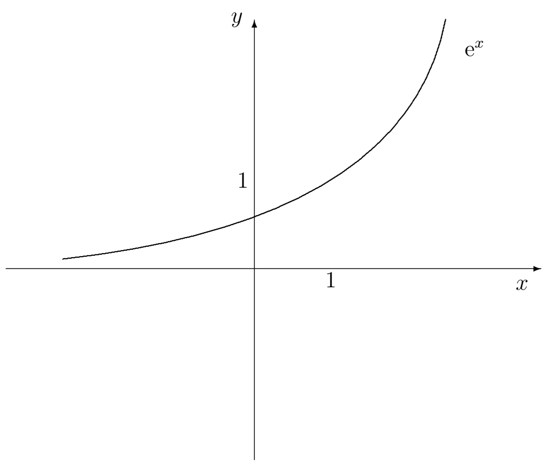
   (c) $g(x) = 9 - x^2$, domain $\mathbb{R}^+$

Worksheet 4.9 Exponentials and Logarithms

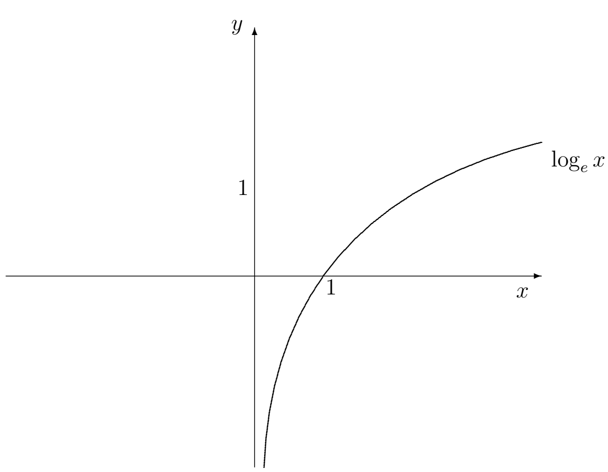
Section 1 Introduction

We have met exponentials and logarithms before in previous modules. Before working through this worksheet you will want to refresh your knowledge of logarithms and exponentials by working through worksheet 2.7 from module 2. In this worksheet we will consider the special relationship logarithms and exponentials have with each other.

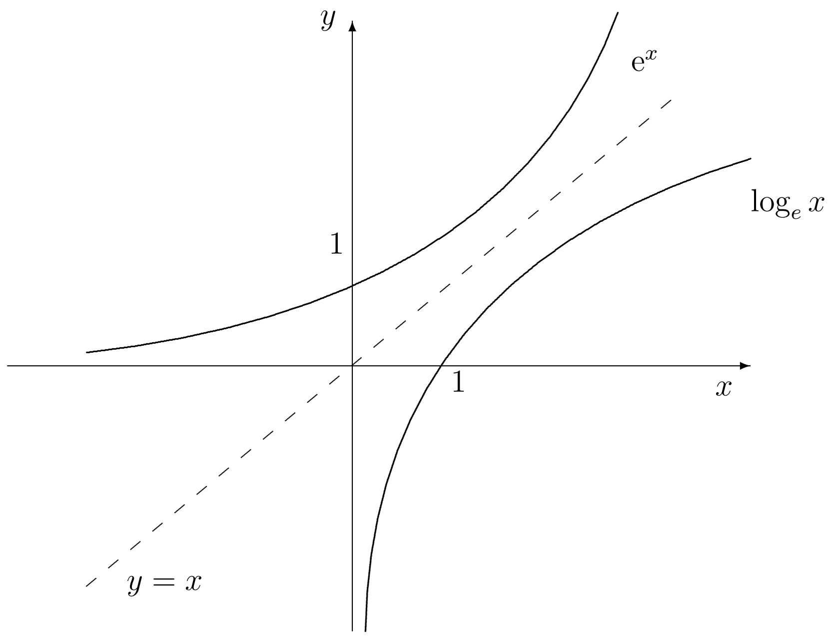
The exponential function e^x looks like



and is a function which grows (or decays) rapidly. The log function $\log_e x$ looks like



We define the log function as the inverse of the exponential, so logs and exponentials "undo" each other. When we sketch their graphs we can easily see the property of inverse functions that they are symmetrical about $y = x$



More formally $f(x) = \mathrm{e}^x$ and $f^{-1}(x) = \log_e x$ whence

$$(f \circ f^{-1})(x) = \mathrm{e}^{\log x} = x \quad \text{and} \quad (f^{-1} \circ f)(x) = log_e(\mathrm{e}^x) = x.$$

Exercises:

1. Use the following exercises to revise your knowledge of exponentials and logarithms.
   (a) Simplify $(4x^3y^{-1})^{\frac{1}{2}} \times \left(\frac{9x^4}{y^{\frac{2}{7}}}\right)^{\frac{5}{2}}$
   (b) Simplify $3^{2x} \times 27^x$
   (c) Solve $8 \times \left(\frac{1}{16}\right)^x = 4^{x+1} \times \sqrt{32}$
   (d) Find the exact value of $\log_5 \left(\frac{1}{\sqrt{125}}\right)$
   (e) Simplify $\log x^3 - \log y + 2\log\left(\frac{y}{x}\right)$
   (f) Solve $\log 4x^4 - 2\log 2x = \log(x+2)$
2. What is the domain and range of $y = \log_5 x$. Write down its inverse function and the corresponding domain and range.

3. Consider the exponential function $f(x) = b^x$ (where $b > 0$). For the following values of b sketch the corresponding exponential functions on the same graph.

   (a) $b = 2$
   (b) $b = \frac{1}{2}$
   (c) $b = 10$

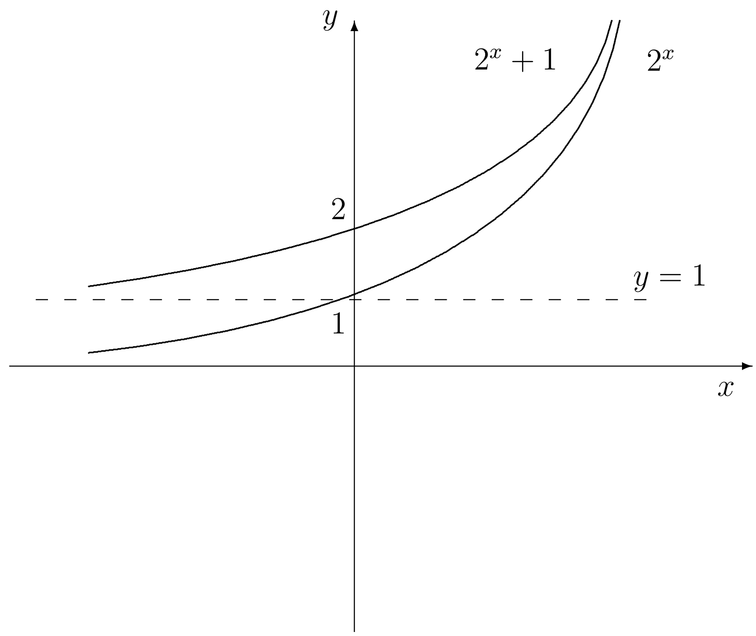
4. For each of the exponentials $f(x)$ you drew in question 3, find its inverse $f^{-1}(x)$ and on a new graph sketch both the function and the inverse you've found.

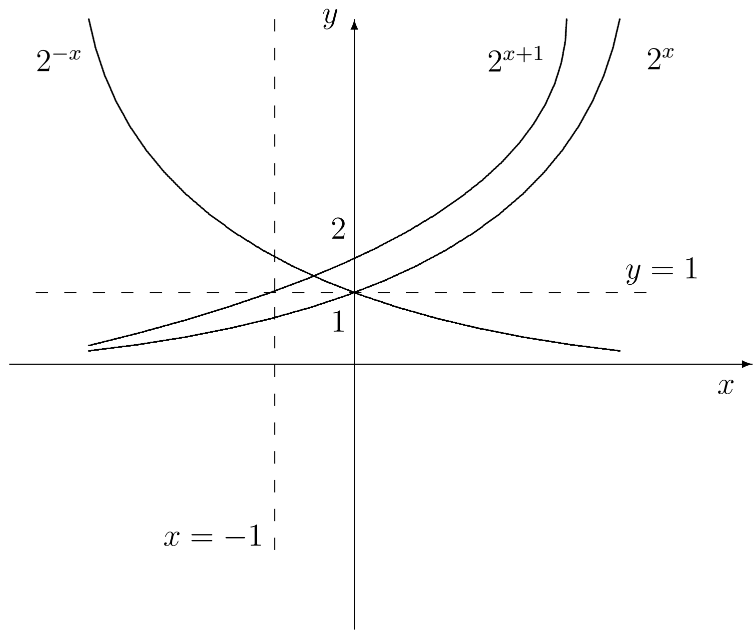
Section 2 MODIFYING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In the same way that we modified functions in Chapter 4.9, we can also easily sketch graphs of modified exponential and logarithmic functions.

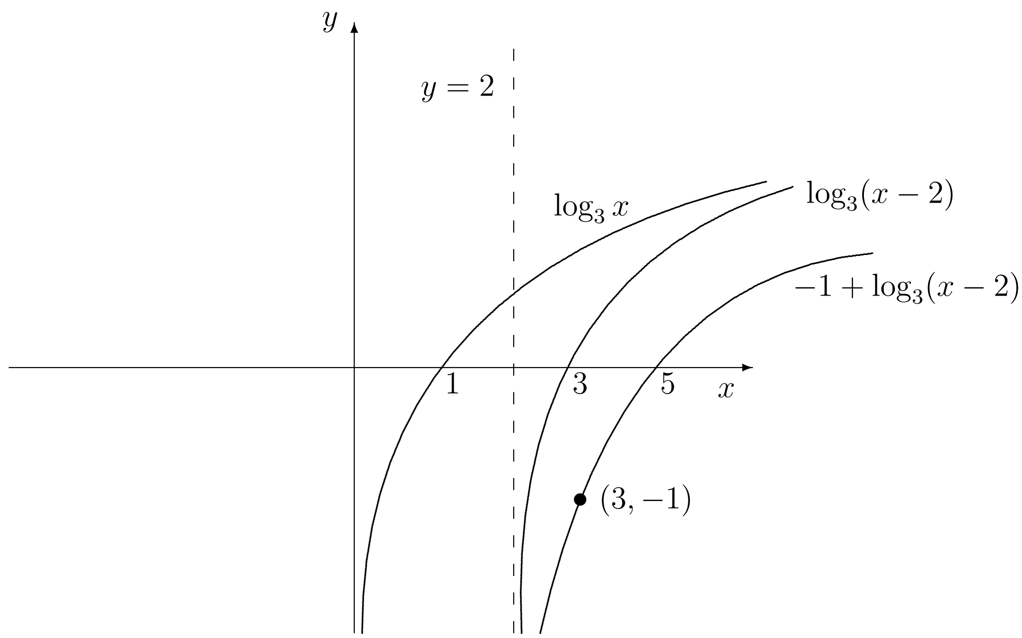
Example 1 : Sketch the graph of $y = 2^x$ and the modifications $y = 2^x + 1, y = 2^{x+1}$ and $y = 2^{-x}$.

The function $y = 2^x$ is essentially the same as $y = \mathrm{e}^x$, just not quite as steep, since $\mathrm{e} \approx 2.71828$ Notice that $y = 2^x + 1$ is a vertical sift up by 1 unit and that $y = 2^{x+1}$ is a horizontal shift to the left by 1 unit. For $y = 2^{-x}$ what happens for negative x values in this graph is what happened for the positive x values in the original graph. So this is just a reflection about the y-axis.





Example 2 : Sketch the graph $y = \log_3 x$ and $y = -1 + \log_3(x - 2)$ on the same axis.
$\log_3 x$ has the same shape as $\log_e x$, increasing at a slightly slower rate. To modify the function we shift it to the right by 2, and subtract 1.



Exercises:

1. Sketch the following exponential functions.

   (a) $y = 3^x$
   (b) $y = 3^{x+1}$
   (c) $y = 1 + 3^x$
   (d) $y = 3^{-x}$
   (e) $y = -3^x$
   (f) $y = \frac{3^x}{3}$

2. Sketch the following functions and state the domain of each.

   (a) $y = \log(x + 3)$
   (b) $y = 1 + \log(x + 3)$
   (c) $y = \log|x + 3|$
   (d) $y = |\log(x + 3)|$

3. Find the domain of the following functions.

   (a) $y = \log(x - 5)$
   (b) $y = \log\sqrt{x - 5}$
   (c) $y = \log|x - 5|$
   (d) $y = \log\sqrt{x^2 - 4}$

Section 3 Exponentials and Logarithms as Inverse Functions

Since the exponential and logarithm functions are inverses we can use the same methods in the previous worksheet to find inverses.

Example 1 : Find the inverse of $f(x) = \log(x + 1)$.
Let $y = \log(x + 1)$. Write $x = \log(y + 1)$ and rearrange

$$\begin{aligned} e^x &= y + 1 \\ y &= e^x - 1, \end{aligned}$$

so

$$f^{-1}(x) = e^x - 1.$$

Example 2 : Find the inverse of $f(x) = \log(x-3)$.
Let $y = \log(x-3)$. Write $x = \log(y-3)$ and rearrange

$$\begin{aligned} e^x &= y - 3 \\ y &= e^x + 3, \end{aligned}$$

so

$$f^{-1}(x) = e^x + 3.$$

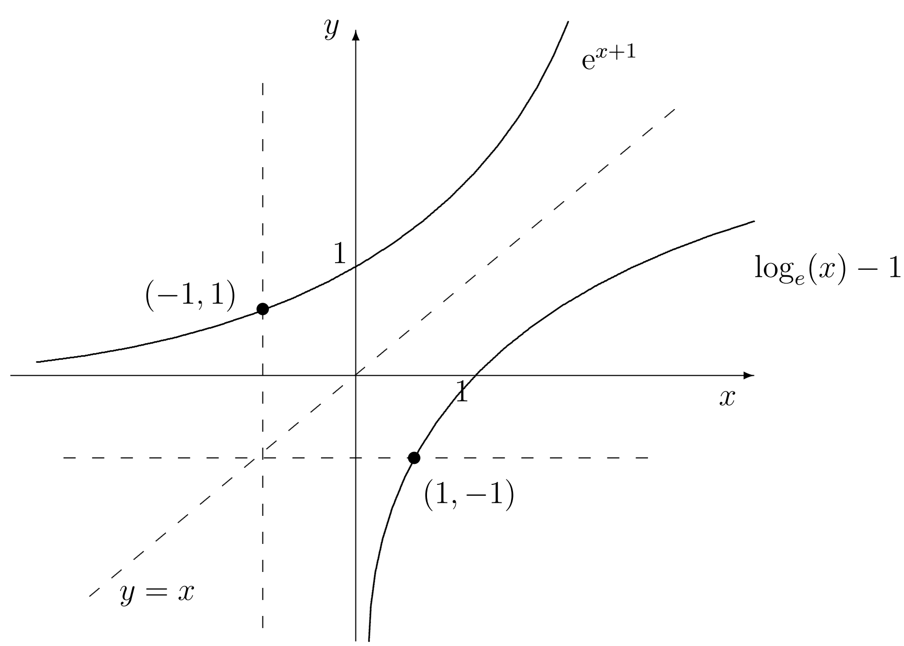
Example 3 : Find the inverse of $g(x) = e^{x+1}$.
Let $y = e^{x+1}$. Write $x = e^{y+1}$ and rearrange

$$\begin{aligned} \log x &= y + 1 \\ y &= \log(x) - 1, \end{aligned}$$

so

$$g^{-1}(x) = \log(x) - 1.$$

- Note: $g : \mathbb{R} \longrightarrow [0, \infty)$ and $g^{-1} : [0, \infty) \longrightarrow \mathbb{R}$.
- We can see the graphical property of inverse functions by sketching both g and g^{-1} on the same diagram:



Example 4 : Find the inverse of $g(x) = \mathrm{e}^{2x-3} + 2$.

Let $y = \mathrm{e}^{2x-3} + 2$. Write $x = \mathrm{e}^{2y-3} + 2$ and rearrange

$$\begin{aligned} 2y - 3 &= \log(x-2) \\ 2y &= \log(x-2) + 3 \\ y &= \frac{\log(x-2)+3}{2}, \end{aligned}$$

so

$$g^{-1}(x) = \frac{\log(x-2)+3}{2}.$$

Exercises:

1. Consider $f(x) = e^{x-2}$.
   (a) Find the inverse of $f(x)$.
   (b) Verify that this is indeed the inverse of $f(x)$ by considering $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$.
   (c) Sketch $f(x)$ and $f^{-1}(x)$ on the same graph. What do you notice?
2. Find the inverse of the following.
   (a) $y = e^{\frac{1}{x}}$
   (b) $y = \log(3 - x)$
   (c) $y = 2 - \log(x + 1)$
3. Find the inverses of the following functions (with given domain) and specify the domains of the corresponding inverse functions.
   (a) $f(x) = e^{2x}$, domain $[1, \infty)$
   (b) $f(x) = \log(2x + 4)$, domain $(-2, \infty)$
4. Explain why $e^{\log(-1)} \neq -1$.

Exercises for Worksheet 4.9

1. Find the domains and ranges of:
   (a) $\log(2x+3)$
   (b) $\log(e^x - 1)$
   (c) e^{2x-1}
   (d) $\log(\sqrt{2x-1}+3)$
2. Sketch the following functions.
   (a) $y = e^{x+4} - 5$
   (b) $y = \log(x-3)$ and $y = \log x - 3$
   (c) $y = \log_e(-x)$
   (d) $y = e^{-x+1}$
   (e) $y = \frac{1}{e^x - 2}$
3. Sketch the following functions.
   (a) $y = |2\log(x-3)|$
   (b) $y = |3e^{x+5} - 7|$
   (c) $y = \left|\frac{1}{e^x-2}\right|$
4. Find the inverses of the following functions
   (a) $y = e^{x-2}$
   (b) $y = e^{x+4} - 5$
   (c) $y = \log_e x - 2$
   (d) $y = \log_e(-x)$
   (e) $y = 3 + \log_2(x+4)$
   (f) $y = 2^{-x+1}$
5. Let $g(x) = \log x$ and $h(x) = x - 2$.
   (a) Find $k(x) = (g \circ h)(x)$.
   (b) Find the domain of $k(x)$.
   (c) Find the range of $k(x)$.
   (d) Find $k^{-1}(y)$.
   (e) Sketch $k(x)$ and $k^{-1}(y)$ on the same plane.
6. (Harder) Find the inverse of $f(x) = \log|x+3|$.

Worksheet 4.10 Sigma Notation

Section 1 Factorial notation

Factorial notation is a shorthand way of writing the product of the first n positive integers. That is for any positive integer n, the notation $n!$ (which is read as 'n factorial') is defined to be

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

In addition, we define $0! = 1$.

Example 3 : What is 4!?

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

Example 4 : What is 5!?

$$5! = 5 \times 4! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Example 5 : Evaluate $\frac{7!}{5!}$.

$$\frac{7!}{5!} = \frac{7 \times 6 \times 5!}{5!} = 7 \times 6 = 42$$

Exercises:

1. Evaluate 9!
2. Evaluate $\frac{5!}{2!3!}$

Section 2 Introduction to sigma notation

Sigma notation is used as a convenient shorthand notation for the summation of terms.

Example 1 : We write

$$\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5.$$

Here the symbol Σ (sigma) indicates a sum. The numbers at the top and bottom of sigma are called boundaries and tell us what numbers we substitute in to the expression for the terms in our sum. What comes after the sigma is an algebraic expression representing terms in the sum. In the example above, n is a variable and represents the terms in our sum.

Example 2 :

$$\sum_{n=1}^{5} n^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3.$$

Example 3 :

$$\sum_{n=3}^{5} n^3 = 3^3 + 4^3 + 5^3.$$

Example 4 :

$$\sum_{n=1}^{4} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

Note that we have $\sum_{n=1}^{5} n = \sum_{i=1}^{5} i$. The n and the i just play the role of dummy variables.

We can also work the other way. Sometimes our sum has a pattern which enables us to write the sum usnig sigma notation.

Example 5 : Write the expression $3 + 6 + 9 + 12 + \cdots + 60$ in sigma notation.

- notice that we are adding multiples of 3;
- so we can write this sum as $\sum_{n=1}^{30} 3n$.

Example 6 : Write the expression $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \cdots + \frac{1}{3n+1}$ in sigma notation.

- notice that we are adding fractions with a numerator of 1 and denominators starting with 1 in the first term and then increasing by 3 in each subsequent term;
- i.e. the denominator can be represented by $3k + 1$ for $k = 0, 1, \ldots, n$;

- so we can write the sum as $\displaystyle\sum_{k=0}^{n} \frac{1}{3k+1}$.

We can also use sigma notation when we have variables in our terms.

Example 7 : Write the expression $3x + 6x^2 + 9x^3 + 12x^4 + \cdots + 60x^{20}$ in sigma notation.

- note from Example 5 the numbers are multiples of 3 and can be represented by $3n$ where $n = 1, 2, \ldots, 20$;
- we also have powers of x which increase by 1 in each subsequent term;
- so we can write this sum as $\displaystyle\sum_{n=1}^{20} 3nx^n$.

The numbers in front of the variables are called coefficients. In Example 7 the coefficients of x is 3 and the coefficient of x^2 is 6.

Example 8 : Write the expression $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!}$ in sigma notation.

- here the powers of x are even numbers which can be represented by $2k$ for $k = 0, 1, \ldots, n$;
- the denominators are also even numbers but with factorials;
- so we can write this sum as $\displaystyle\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$.

Exercises:

1. Write out each of the following sums.

(a) $\displaystyle\sum_{n=1}^{6} n^4$

(b) $\displaystyle\sum_{k=3}^{7} \frac{k+1}{k}$

(c) $\displaystyle\sum_{i=2}^{n} (2i-1)$

(d) $\displaystyle\sum_{k=0}^{n} 2^{k+1}x^k$

(e) $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k x^k}{2k+1}$

2. Express each of these sums using sigma notation.

(a) $1+4+9+16+25+36$

(b) $3-5+7-9+11-13+15$

(c) $\frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}$

(d) $\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots+\frac{n+1}{n+2}$

(e) $2-2^2+2^3-2^4+\cdots+2^{2n+1}$

(f) $2x^3+4x^5+6x^7+\cdots+30x^{31}$

(g) $x-x^2+\frac{x^3}{2!}-\frac{x^4}{3!}+\frac{x^5}{4!}-\frac{x^6}{5!}$

(h) $3x+7x^2+11x^3+15x^4+19x^5+23x^6$

(i) $8x^4+10x^5+12x^6+\cdots+(2n+2)x^{n+1}$

(j) $12x^4+20x^5+30x^6+\cdots+n(n-1)x^n$

(k) $\frac{x^3}{2}-\frac{x^5}{3}+\frac{x^7}{4}-\frac{x^9}{5}+\cdots+\frac{x^{199}}{100}$

Section 3 Finding coefficients

Sigma notation is a useful way to express the sum of a large number of terms. When we want to find particular terms or coefficients, we don't always have to expand the whole expression to find it.

Example 1 : Find the coefficient of x^4 in $\displaystyle\sum_{k=0}^{8}(4k+3)x^k$.

- the terms in this sum look like $(4k+3)x^k$;
- the terms with x^4 occurs when $k=4$ i.e. $(4(4)+3)x^4 = 19x^4$;
- the coefficients of x^4 is 19.

Example 2 : Find the coefficient of x^7 in $\displaystyle\sum_{k=0}^{8}(4k+3)x^{k+2}$.

- a typical term is of the form $(4k+3)x^{k+2}$;

- the term with x^7 occurs when $k+2=7$, i.e. $k=5$;
- we have $(4(5)+3)x^{5+2} = 23x^7$;
- the coefficients of x^7 is 23.

<u>Example 3</u> : Find the coefficient of x^2 in $(3+x)\sum_{k=0}^{8}(4k+3)x^k$.

- we can think of this as $3\sum_{k=0}^{8}(4k+3)x^k + x\sum_{k=0}^{8}(4k+3)x^k$;
- the term with x^2 can be obtained by taking $k=2$ from the first part of this expression to get $3(4(2)+3)x^2 = 33x^2$ and then taking $k=1$ from the second part of this expression to get $x(4(1)+3)x^1 = 7x^2$;
- combining these we get $33x^2 + 7x^2 = 40x^2$;
- so the coefficient of x^2 is 40.

Exercises:

1. Find the coefficients of x^2 and x^6 in the following.

   (a) $\displaystyle\sum_{r=0}^{10}\frac{r+1}{r!}x^r$

   (b) $\displaystyle\sum_{k=3}^{15}k(k+1)x^{k-2}$

   (c) $\displaystyle\sum_{n=0}^{20}\frac{(-1)^n x^{4n+2}}{n+3}$

   (d) $\displaystyle(3+2x)\sum_{k=0}^{8}(k+1)x^k$

   (e) $\displaystyle(1-x)\sum_{k=0}^{7}\frac{x^{k+1}}{k!}$

   (f) $\displaystyle(x+x^2)\sum_{k=0}^{15}(2k+1)x^k$

Exercises for Worksheet 4.10

1. Write out each of the following sums.

   (a) $\displaystyle\sum_{r=0}^{7} r^2(-x)^r$ (b) $\displaystyle\sum_{k=3}^{8} \frac{k-1}{k+1}x^{2k}$ (c) $\displaystyle\sum_{k=1}^{n+1} k(k-1)x^{3k}$

2. Write each of the following series in sigma notation.

   (a) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$
   (b) $7 - 10 + 13 - 16 + \cdots + 31$
   (c) $4^2 + 5^2 + 6^2 + 7^2 + \cdots + (n+2)^2$
   (d) $\frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \cdots + \frac{1}{(n+5)!}$
   (e) $3x^2 + 6x^4 + 9x^6 + 12x^8 + \cdots + 36x^{24}$
   (f) $x^7 + \frac{x^9}{1!} + \frac{x^{11}}{2!} + \frac{x^{13}}{3!} + \cdots + \frac{x^{31}}{12!}$
   (g) $5x - 9x^2 + 13x^3 - 17x^4 + \cdots - 41x^{10}$
   (h) $\frac{5}{2}x^3 + 3x^4 + \frac{7}{2}x^5 + 4x^6 + \cdots + \frac{n}{2}x^{n-2}$
   (i) $6x^{12} + 7x^{14} + 8x^{16} + 9x^{18} + \cdots + (n+1)x^{2n+2}$

3. Find the coefficient of x, x^3 and x^7 in the following expressions.

   (a) $\displaystyle\sum_{k=3}^{n} \frac{(-1)^k x^{k-2}}{(4k-1)!}$
   (b) $\displaystyle\sum_{k=1}^{n} \frac{(3x)^{k-3}}{(k+1)^2}$
   (c) $\displaystyle(2+x)\sum_{k=0}^{n} \frac{k-1}{k+1}x^k$
   (d) $\displaystyle(x-x^2)\sum_{k=0}^{n} \frac{(-1)^{k+1}}{3k!}x^k$
   (e) $\displaystyle(5+x^2)\sum_{k=1}^{n} \frac{k}{(2k+1)!}x^{k-2}$

4. Simplify the following expressions.

   (a) $\displaystyle\sum_{k=0}^{n} k^2 - (k+1)^2$
   (b) $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k}\right)$

Worksheet 4.11 Counting

Section 1 Introduction

When looking at situations involving counting it is often not practical to count things individually. Instead techniques have been developed to help us count efficiently and accurately. In particular this chapter looks at permutations and combinations.

Firstly however we must look at the Fundamental Principle of Counting (sometimes referred to as the multiplication rule) which states:

If there are m ways of doing one thing and n ways of doing another, then there are $m \times n$ ways of doing the first thing followed by the second.

This rule can best be understood by looking at an example.

Example 1 : There are 3 t-shirts and 2 pairs of jeans in the cupboard. How many possible outfits are there?

For each t-shirt there are 2 possibilities of jeans. All together we have three lots of these two possibilities, or, $3 \times 2 = 6$.

Example 2 : An ice-cream shop offers 3 types of cones and 5 different flavours of ice-cream. How many possible ice-cream cone combinations are there?

For each of the 3 cones there are 5 possible toppings so altogether there are $3 \times 5 = 15$ possible ice-cream cone combinations.

Exercises:

1. The local pizzeria offers a choice of 2 pizzas - supreme or vegetarian, 3 sides - chips, salad or coleslaw, and 4 drinks - juice, coke, ginger beer or water. For dinner I decide to have 1 pizza, 1 side, and 1 drink. How many possible meals do I have to choose from?
2. How many different car number plates can be made if each is to display 3 letters followed by 3 numbers?
3. Your friend wants to perform a magic trick and asks you to draw 2 cars from a standard deck of 52. The first card you draw must be placed face down and the second placed face up on the table. How many ways are there of drawing the 2 cards?

Section 2 Counting Techniques

For the many circumstances where we need to count the number of outcomes there are two different counting situations - permutations and combinations. A permutation is an arrangement where the order of selection matters. A combination is an arrangement where the order of selection doesn't matter.

Example 1 : The number of ways of arranging 5 people in a line. There are 5 choices for the first spot, 4 choices for the second and so on, so we have $5\times4\times3\times2\times1 = 5!$. Therefore the number of ways of arranging n things into n places is $n!$.

Example 2 : To arrange 3 people out of 5 in a line there are $5 \times 4 \times 3 = \frac{5!}{2!}$ ways. These are all permutations, the number of ways of choosing k things out of n where the order matters is

$$= \frac{n!}{(n-k)!} = {}^n\mathrm{P}_k.$$

How about choosing 3 out of 5 people to put on a committee? This time, it doesn't matter whether you are chosen first, second or third, you are still on the committee. There are $5\times4\times3$ ways of choosing 3 people in order. However since order doesn't matter we have over counted, and therefore we need to divide by the number of ways of arranging the three people that are chosen, 3!. So the number of ways of choosing this committee is the number of ways of choosing with order, divided by $3 \times 2 \times 1$.

$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = \frac{5!}{2!3!}.$$

This is a combination, the number of ways of choosing k things from n where order does **NOT** matter

$$= \frac{n!}{k!(n-k)!} = {}^n\mathrm{C}_k.$$

Definition 1 : A permutation is the number of ways of choosing k things out of a possible n, where the order that they are chosen matters and is notated ${}^n\mathrm{P}_k$.

$${}^n\mathrm{P}_k = n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!}.$$

Definition 2 : A combination is the number of ways of choosing k things out of a possible n, where the order that they are chosen does not matter and is notated ${}^{n}\mathrm{C}_{k}$.

$$ {}^{n}\mathrm{C}_{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)\dots1} = \frac{n!}{(n-k)!k!}. $$

These definitions are best illustrated by examples which we'll cover below and in the next 2 sections.

Example 3 :

(a) A teacher wants to randomly choose 5 people from the class of 30 to help out at the open day BBQ. In how many ways can this be done?

(b) A teacher wants to award prizes for 1st, 2nd, 3rd, 4th and 5th in the class of 30. In how many ways can the prizes be awarded (assume no two students tie)?

Answers:

(a) In the first case, the 5 people are chosen, and it doesn't matter whether a person is chosen first, second or fifth, they all receive the same extra work, and so there are

$$ {}^{30}\mathrm{C}_{5} = \frac{30 \times 29 \times 28 \times 27 \times 26}{5 \times 4 \times 3 \times 2 \times 1} = 142,506 $$

ways to choose the students with those conditions.

(b) Here the order that the student are chosen does matter, and so there are

$$ {}^{30}\mathrm{P}_{5} = 30 \times 29 \times 28 \times 27 \times 26 = 17,100,720 $$

ways to choose the students with those conditions. This is a significantly more as expected.

Notice the short way of calculating ${}^{30}\mathrm{P}_{5}$ and ${}^{30}\mathrm{C}_{5}$ applies to all such calculations. To find ${}^{30}\mathrm{P}_{5}$ you can just multiply 5 consecutive descending numbers together, starting at 30. Similarly calculating ${}^{30}\mathrm{C}_{5}$ can be done by multiplying 5 consecutive descending numbers together, starting at 30, and then dividing by the product of 1 through 5.

Exercises:

1. Calculate ${}^{10}\mathrm{P}_4$ and ${}^{10}\mathrm{C}_4$.
2. Calculate ${}^{7}\mathrm{P}_5$ and ${}^{7}\mathrm{C}_2$.
3. In an 18 team league, how many ways can the 8 teams for the finals be decided? In how many ways can the first 4 positions be decided?
4. A child wants to draw a picture using only three different colours from a set containing twelve different colour pencils. In how many ways can the colours be chosen?
5. In the front of a building there are three doors each to be painted a different colour from twelve different available colours. How many colour arrangements for the doors are there?

Section 3 PERMUTATIONS

Example 1 :

(a) How many four digit numbers can be formed using only the digits $1, 2, 3, 4, 5, 6$?

(b) How many four digit numbers from (a) have no repeated digits?

(c) How many four digit numbers from (b) are greater than 5000?

Answers:

(a) There are 6 possible digits for each of the four places in the number, so there are $6 \times 6 \times 6 \times 6 = 6^4 = 1296$ of these numbers.

(b) There are 6 digits for the first place, and then only 5 digits for the second and so on. So there are $6 \times 5 \times 4 \times 3 = 360$ such numbers.

(c) There are 2 choices for the first digit (a 5 or a 6), then 5 choices, 4 choices and 3 choices respectively for the remaining digits. So there are $2 \times 5 \times 4 \times 3 = 120$ such numbers.

Example 2 : Three adults and five children are seated randomly in a row.

(a) In how many ways can this be done?

(b) In how many ways can this be done if the three adults are seated together?

(c) In how many ways can this be done if the three adults are seated together and the five children are also seated together.

Answers:

(a) There are 8! ways of arranging 8 people in a row.

(b) There are 3! ways of arranging the adults. We now need to arrange 6 objects (1 group of adults and 5 individual children) in a row. Therefore the answer is $3! \times 6!$ ways.

(c) There are 3! ways of arranging the adults, 5! ways of arranging the children, and 2! ways of arranging the 2 groups. So the answer is $3! \times 5! \times 2!$.

Permutations with Repeated Objects

Example 3 : How many arrangements of the letters of the word IRRIGATION are there?

Answer: There are 10 letters. If the letters were all different there would be 10! arrangements. However there are three I's and two R's and so we need to divide by $3! \times 2!$. Therefore the answer is $\frac{10!}{3!2!}$.

Example 4 : In how many ways can we rearrange the letters in "MATHS IS FUN"

(a) with no restrictions?

(b) if the first and last letter must be vowels?

Answers:

(a) There are 10 letters to rearrange. Two of them are S's and so that we do not over count we need to divide by the number of ways of arranging the S', so there are $\frac{10!}{2!}$ ways to arrange these letters.

(b) In the second instance, we first select the vowels and there are 3×2 ways of the selecting the first and last as vowels. Then we have 8 letters left with 2 S's repeating so there are $\frac{8!}{2!}$ ways to do this. Hence there are $3 \times 2 \times \frac{8!}{2!}$ ways of arranging these letters so that the first and last are vowels.

Exercises:

1. How many 4 digit numbers can be formed from the digits 1,2,3,4 if

   (a) repetitions are allowed?

(b) repetitions are not allowed?

2. How many 3 digit arrangements can be formed from the digits 0,1,2,3,4,5,6,7 and 8?

3. Seven people are to occupy consecutive seats in a theatre. In how many ways can this be done if
(a) there are no restrictions?
(b) two people A and B sit at opposite ends of the row of seven seats?
(c) two people A and B sit together?
(d) two people A and B do not sit together?

4. How many arrangements of the letters in the following words are possible?
(a) SYDNEY (b) GEOMETRY (c) EXCELLENCE

Section 4 Combinations

Example 1 : A student must select 6 subjects. In how many ways can they do that if there are 13 subjects and 1 is compulsory?

Since one subject is compulsory the student must select 5 subjects from 12, there are

$$^{12}\mathrm{C}_5 = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 792,$$

ways to do this.

Example 2 : In a lottery you select 6 numbers out of 40, how many ways are there to do this?

You are selecting 6 things from 40 with order not mattering, thus there are

$$^{40}\mathrm{C}_6 = \frac{40 \times 39 \times 38 \times 37 \times 36}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 3,838,380\text{ways}.$$

Example 3 : You want to choose a committee of 5 people from 7 men and 8 women.

(a) How many ways can this be done?

(b) How many ways can this be done if you want a majority of women on the committee?

In (a) it doesn't matter if the 5 people are men or women. So we are choosing 5 people from 15 and there are $^{15}C_5$ ways of doing this.

In (b) we need either 3 women (and 2 men) or 4 women (and 1 man) or 5 women. So there are $^8C_3 \times {}^7C_2 + {}^8C_4 \times {}^7C_1 + {}^8C_5$ ways of doing this.

Exercises:

1. In how many ways can you choose 2 chocolates from a bag containing 6 different chocolates?

2. Twelve dots are spaced equally around a circle. How many different triangles can be formed by joining dots?

3. Tickets are numbered from 1 to 25. 6 tickets are chosen. In how many ways can this be done if the selection contains
   (a) all odd numbers?
   (b) 3 odd numbers and 3 even numbers?
   (c) the numbers 1 and 2?

Exercises for Worksheet 4.11

1. Decide whether or not order of selection is important and then calculate the following.
    (a) How many different sets of three colours can be selected from the colours red, orange, yellow, green, blue, and violet?
    (b) In how many ways can a team of five basketball players be selected from 8 girls?
    (c) A race has 8 runners. In how many ways can the first three places be decided?
    (d) A secretary has nine letters and only five stamps. How many ways can he select the letters for posting?
2. How many different possible "full house" (one pair, one three of a kind) hands are there in 5 card poker?
3. We choose 12 cards from the usual deck of 52 playing cards.
    (a) How many different ways can this be done?
    (b) How many ways can it be done if they must all come from the same suit?
    (c) How many ways can it be done if we need exactly 3 kings and 3 queens?
    (d) How many ways can it be done if all cards must have different face values?
4. There are 10 boys and 11 girls at a school.
    (a) How many different ways can the boys each choose a girl to take to the formal?
    (b) One of the girls doesn't want to go to the formal, how many ways are there to make this choice now?
5. A town of 30 people is to choose a committee of 3 to represent them, how many different ways can this be done? How many ways can it be done if one person is to be the chairperson, one the treasurer and one the secretary?
6. In a certain electorate there are 6 candidates: labor, liberal, greens, and three independents. Their names are to be placed in random order on the ballot paper. In how many ways can this be done if
    (a) the labor candidate comes first?
    (b) the liberal candidate comes first?
    (c) the three independent candidates are together?
7. From a class of 30 students, five students are to be selected to complete a survey. In how many ways can the choice be made?

8. A 4 digit password is to be formed from the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9. How many different passwords are possible
   (a) if the digits are repeated?
   (b) if there are no repeated digits?
9. How many different arrangements of the word ELLIPSE are possible if
   (a) there are no restrictions?
   (b) the arrangement starts with S?
   (c) both L's are together?
   (d) the letters are in alphabetical order?
10. Seven people sit in a circle. How many ways can this be done if
   (a) there are no restrictions?
   (b) two people A and B sit together?
   (c) three people A, B and C sit together?
11. A committee of 5 is to be chosen from 4 men and 6 women. In how many ways can this be done if
   (a) there are no restrictions?
   (b) the committee consists of women only?
   (c) there is at least one man?
   (d) there is a majority of women?
12. 5 cards are to be chosen from a standard 52-card deck. In how many ways can this be done if
   (a) all of the cards are clubs?
   (b) all of the cards are of the same suit?
   (c) there are three clubs and two spades?
   (d) there are three of one suit and two of another?
13. A bag contains 5 red, 6 blue and 4 yellow marbles. Three are drawn out at random. In how many ways can they be drawn so that
   (a) are all blue?
   (b) are all the same colour?
   (c) are all different colours?

14. In order to be photographed 10 people stand in two rows of 5, one in front of the other. In how many ways can this be done if

    (a) there are no restrictions?

    (b) A and B are to be in the front row?

    (c) A and B are in different rows?

15. In how many ways can you choose 10 people out of 30 to sit on a bench in order, if two particular people must be selected and seated together?

16. You are to pick three singers for the Superbowl, one to sing before the game, one to sing at the end of the game, and one to sing at halftime. If there are 12 applicants, how many different possibilities are there if each singer can only sing once?

Worksheet 4.12 The Binomial Theorem

Section 1 Binomial Coefficients and Pascal's Triangle

We wish to be able to expand an expression of the form $(a+b)^n$. We can do this easily for $n = 2$, but what about a large n? It would be tedious to manually multiply $(a+b)$ by itself 10 times, say. There are two methods of expanding an expression of this type without doing all the multiplications involved. The first method we will look at is called Pascal's triangle. The first 5 rows of Pascal's triangle are shown:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Pascal's triangle is particularly useful when dealing with small n. The triangle is easy to remember as each entry is the sum of the two right and left entries on the line above, and the sides are always one. Thus for the second entry of line five we get

$$\begin{array}{ccc} 1 & & 3 \\ & \searrow\swarrow & \\ & 4 & \end{array}$$

The entries in Pascal's triangle tell us the coefficients when we expand expansions like $(a+b)^n$. Look at $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2 = 1a^2 + 2ab + 1b^2$. Notice that the coefficients in front of each term 1 2 1 appear in the 3rd line of Pascal's triangle.

What about $(a+b)^3 = (a+b)(a+b)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$. Notice that the coefficients come from the 4th line of Pascal's triangle as 1 3 3 1.

You may also notice that as the power of a decreases, the power of b increases. This phenomenon can be generalised.

For the expansion of $(a+b)^n$ we need the $(n+1)^{th}$ line of Pascal's triangle. The first term in the line has an $a^n b^0$ and then the powers of a decrease and the powers of b increase, so the second term will be $a^{n-1}b^1$ and so on, until the last tern has $a^0 b^n$.

Example 1 : Expand $(a+b)^5$. We use the 6th line of Pascal's triangle to obtain

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Notice that the powers of a and b in each term always add to n, where n is the power to which (a+b) is raised. In the above example we can see the power of a and b in each term always adds to 5.

Example 2 : Expand $(1+x)^4$. Using the fifth row of Pascal's triangle:

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

If you wish to use Pascal's triangle on an expansion of the form $(ax+b)^n$, then some care is needed. The $(n+1)$th row is the row we need, and the 1st term in the row is the coefficient of $(ax)^n b^0$. The second term in the row is the coefficient of $(ax)^{n-1}b^1$. The last term in the row - the $(n+1)$th term - is the coefficient of $(ax)^0 b^n$. Care should be taken when minus signs are involved.

Example 3 : Expand the expression $(ax+b)^3$. The 4th row of Pascal's triangle is 1 3 3 1. So

$$\begin{aligned}(ax+b)^3 &= (ax)^3 + 3(ax)^2 b + 3(ax)^1 b^2 + b^3 \\ &= a^3x^3 + 3a^2bx^2 + 3ab^2x + b^3\end{aligned}$$

Notice that the powers of a and b in each term always add to give the power of the expansion. This is always the case.

Example 4 : Expand the expression $(2x-3)^3$. Using the 4th row of pascals triangle:

$$\begin{aligned}(2x-3)^3 &= (2x)^3 + 3(2x)^2(-3) + 3(2x)^1(-3)^2 + (-3)^3 \\ &= 8x^3 - 36x^2 + 54x - 27\end{aligned}$$

Example 5 : What is the coefficient of x^2 in the expansion of $(x+2)^5$? The 6th line of Pascal's triangle is 1 5 10 10 5 1. Remember when we expand $(a+b)^5$ the powers of a decrease as the powers of b increase and their powers will add to 5. In this example $a = x$ and $b = 2$. The term with an x^2 in it will be of the form a^2b^3 - the fourth term in the expansion. Hence we use the fourth number from our line in Pascal's triangle as the coefficient in front of this term. So the term will look like $10a^2b^3$. Since $a = x$ and $b = 2$ and $2^3 = 8$ we see that $10a^2b^3 = 10x^22^3 = 80x^2$. Thus, the coefficient of x^2 is 80.

Example 6 : Find the constant term (the term that is independent of x) in the expansion of $(x-2)^5$. The constant term is the last term, and is $(-2)^5$. Notice that the minus sign is important.

Example 7 : Find the 4th term in the expansion of $(2x-3)^5$. The 4th term in the 6th line of Pascal's triangle is 10. So the 4th term is

$$10(2x)^2(-3)^3 = -1080x^2$$

The 4th term is $-1080x^2$.

The second method to work out the expansion of an expression like $(ax+b)^n$ uses binomial coefficients. This method is more useful than Pascal's triangle when n is large.

Exercises:

1. Write the first 6 lines of Pascal's triangle.
2. Expand $(x+y)^4$ using Pascal's triangle.
3. What line of Pascal's triangle is $1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1$?
4. Complete this line of Pascal's triangle "$1, 8, 28, 56, 70, 56, \ldots$". Hence also write the next line of Pascal's triangle.
5. Expand $(2a-3)^5$ using Pascal's triangle.

Section 2 Binomial Theorem

Calculating coefficients in binomial functions, $(a+b)^n$, using Pascal's triangle can take a long time for even moderately large n. For example, it might take you a good 10 minutes to calculate the coefficients in $(x+1)^8$. Instead we can use what we know about combinations.

Example 1 : What is the coefficient of x^7 in $(x+1)^{39}$
To answer this, we think of it as a counting question. In the product of 39 copies of $(x+1)$ we need to choose 7 x's, and the order that they are chosen in does not matter. Thus the coefficient of x^7 is ${}^{39}\mathrm{C}_7 = 15,380,937$.

So it turns out that the numbers in Pascals triangle, which we saw were coefficients in binomial expansions, are actually the numbers that come up in combinations, ${}^n\mathrm{C}_k$. For simplicity of writing we will define ${}^n\mathrm{C}_k = \binom{n}{k}$.

Definition 2 : The binomial theorem gives a general formula for expanding all binomial functions:

$$\begin{aligned}(x+y)^n &= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{r} x^{n-r} y^r + \cdots + \binom{n}{n} y^n,\end{aligned}$$

recalling the definition of the sigma notation from Worksheet 4.6.

Example 2 : Expand $(x+y)^8$

$$\begin{aligned}(x+y)^8 &= \binom{8}{0} x^8 + \binom{8}{1} x^7 y + \binom{8}{2} x^6 y^2 + \binom{8}{3} x^5 y^3 + \binom{8}{4} x^4 y^4 + \binom{8}{5} x^3 y^5 \\ &\quad + \binom{8}{6} x^2 y^6 + \binom{8}{7} x y^7 + \binom{8}{8} y^8 \\ &= x^8 + 8x^7 y + 28x^6 y^2 + 56x^5 y^3 + 70x^4 y^4 + 56x^3 y^5 + 28x^2 y^6 + 8xy^7 + y^8\end{aligned}$$

Example 3 : Expand $(2x-3)^5$

$$\begin{aligned}(2x-3)^5 &= \binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4(-3) + \binom{5}{2}(2x)^3(-3)^2 + \binom{5}{3}(2x)^2(-3)^3 \\ &\quad + \binom{5}{4} 2x(-3)^4 + \binom{5}{5}(-3)^5 \\ &= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243\end{aligned}$$

Example 4 : Expand $\left(\frac{1}{x} + 2x^2\right)^4$

$$\begin{aligned}\left(\frac{1}{x} + 2x^2\right)^4 &= \binom{4}{0}\left(\frac{1}{x}\right)^4 + \binom{4}{1}\left(\frac{1}{x}\right)^3 2x^2 + \binom{4}{2}\left(\frac{1}{x}\right)^2 (2x^2)^2 + \binom{4}{3}\left(\frac{1}{x}\right)(2x^2)^3 \\ &\quad + \binom{4}{4}(2x^2)^4 \\ &= \frac{1}{x^4} + \frac{8}{x} + 24x^2 + 32x^5 + 16x^8\end{aligned}$$

Example 5 : Find the coefficient independent of x in

$$\left(x^3 + \frac{2}{x}\right)^{20}.$$

The binomial theorem tells us that

$$\left(x^3+\frac{2}{x}\right)^{20}=\sum_{i=0}^{20}\binom{20}{i}x^{3i}\left(\frac{2}{x}\right)^{20-i}=\sum_{i=0}^{20}\binom{20}{i}x^{3i-(20-i)}2^{20-i}.$$

So the power of x is $4i-20$. We need to set this to zero to have the constant term, so we need $4i-20=0 \to 4i=20 \to i=5$. Thus the coefficient is

$$\binom{20}{5}2^{15}=508,035,072.$$

Exercises:

1. Expand
   (a) $(x^2-1)^4$
   (b) $(x^3-\frac{1}{x^2})^3$
2. Find the coefficients of x, x^2 and x^3 in $(x+2)^5$.
3. Find the coefficients of x, x^2 and x^4 in $(x-2)^7$.
4. Find the coefficients of x and x^{-9} in $\left(2x^3-\frac{3}{x}\right)^7$.
5. Find the exact value of $(1-0.1)^3$ without the use of a calculator. Confirm your answer with a calculator.

Exercises for Worksheet 4.12

1. (a) Show that ${}^5C_2 = {}^5C_3$

   (b) Write down expansions of the following:

   i. $(2x + 3y)^4$

   ii. $\left(a + \dfrac{1}{a}\right)^6$

   iii. $\left(\dfrac{a}{b} - \dfrac{b}{a}\right)^7$

   iv. $(x^2 - 2)^5$

2. (a) Find the coefficients of

   i. x^2 in $\left(x + \dfrac{1}{x}\right)^8$

   ii. a^5b^4 in $\left(3a - \dfrac{b}{3}\right)^9$

   (b) Find the constant terms in

   i. $\left(2x - \dfrac{1}{x^2}\right)^9$

   ii. $\left(2x + \dfrac{1}{x}\right)^{10}$

3. Express $(1.1)^4$ as a binomial of the form $(a + b)^n$, and evaluate it.

Worksheet 4.13 Induction

Mathematical Induction is a method of proof. We use this method to prove certian propositions involving positive integers. Mathematical Induction is based on a property of the natural numbers, $\mathbb{N}$, called the Well Ordering Principle which states that every nonempty subset of positive integers has a least element.

There are two steps in the method:

Step 1: Prove the statement is true at the starting point (usually $n = 1$).

Step 2: Assume the statement is true for n.
Prove the statement is true for $n + 1$ (using the assumption).

Example 1 : Prove $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbb{N}$.

Step 1: [We want to show this is true at the starting point $n = 1$.]

$$\text{LHS} = 1$$
$$\text{RHS} = 1^2 = 1$$

Since LHS=RHS, the statement is true or $n = 1$.

Step 2: Assume the statement is true for n.
i.e. $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$.
[Want to show this is true for $n + 1$.
i.e. Want to show $1 + 3 + 5 + \cdots + (2n - 1) = (n + 1)^2$]

$$\begin{aligned} \text{LHS} &= \underbrace{1 + 3 + 5 + \cdots + (2n - 1)}_{} + (2n + 1) \\ &= \qquad\quad n^2 \qquad\quad + (2n + 1) \qquad \text{(by assumption)} \\ &= n^2 + 2n + 1 \\ &= (n + 1)^2 \\ &= \text{RHS} \end{aligned}$$

So the statement is true for $n + 1$. Hence, the statement is true for all $n \in \mathbb{N}$ by induction. $\square$

Example 2 : Prove $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

Step 1: [We want to show this is true at the starting point $n = 1$.]

$$\text{LHS} = \sum_{k=1}^{n} k^2 = 1^2 = 1$$
$$\text{RHS} = \tfrac{1}{6}1(1+1)(2(1)+1) = 1$$

Since LHS=RHS, the statement is true or $n = 1$.

Step 2: Assume the statement is true for n.

i.e. $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$

[Want to show this is true for $n+1$.

i.e. Want to show $\sum_{k=1}^{n+1} k^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$]

$$\begin{aligned}\text{LHS} &= \sum_{k=1}^{n+1} k^2 \\ &= \underbrace{1^2 + 2^2 + \cdots + n^2} + (n+1)^2 \\ &= \tfrac{1}{6}n(n+1)(2n+1) + (n+1)^2 \qquad \text{(by assumption)} \\ &= \tfrac{1}{6}(n+1)(n(2n+1) + 6(n+1)) \\ &= \tfrac{1}{6}(n+1)(2n^2 + 7n + 6) \\ &= \tfrac{1}{6}(n+1)(n+2)(2n+3) \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n+1$. Hence, the statement is true for all $n \in \mathbb{N}$ by induction. □

Example 3 : Prove $2^n > n^2$ for $n \geq 5$.

Step 1: [We want to show this is true at the starting point $n = 5$.]

$$\text{LHS} = 2^5 = 32$$
$$\text{RHS} = 5^2 = 25$$

Since LHS>RHS, the statement is true or $n = 5$.

Step 2: Assume the statement is true for n, i.e. $2^n > n^2$.

[Want to show this is true for $n+1$. i.e. want to show $2^{n+1} > (n+1)^2$.]

$$\begin{aligned} \text{LHS} &= 2^{n+1} \\ &= 2^n \cdot 2 \\ &> 2n^2 && \text{(by assumption)} \\ &= n^2 + n^2 \\ &= n^2 + 2n + 1 && \text{(since } n^2 > 2n+1 \text{ for } n \geq 5) \\ &= (n+1)^2 \\ &= \text{RHS} \end{aligned}$$

So $2^{n+1} > (n+1)^2$ for $n \geq 5$. i.e. the statement is true for $n+1$ whenever $n \geq 5$. Hence, the statement is true for all $n \geq 5$ by induction. □

Example 4 : Prove that $9^n - 2^n$ is divisible by 7 for all $n \in \mathbb{N}$.

Step 1: [We want to show this is true at the starting point $n = 1$.]

When $n = 1$, we have $9^1 - 2^1 = 7$ which is divisible by 7.

The statement is true for $n = 1$.

Step 2: Assume the statement is true for n.

i.e. Assume $9^n - 2^n$ is divisible by 7.

i.e. Assume $9^n - 2^n = 7m$ for some $m \in \mathbb{Z}$.

[Want to show this is true for $n+1$.

i.e. Want to show $9^{n+1} - 2^{n+1}$ is divisible by 7.]

$$\begin{aligned} 9^{n+1} - 2^{n+1} &= 9 \cdot 9^n - 2 \cdot 2^n \\ &= 9(7m + 2^n) - 2 \cdot 2^n && \text{(by assumption)} \\ &= 7(9m) + 9 \cdot 2^n - 2 \cdot 2^n \\ &= 7(9m) + 7 \cdot 2^n \\ &= 7(9m + 2^n), \end{aligned}$$

which is divisible by 7. So the statement is true for $n+1$. Hence, the statement is true for all $n \in \mathbb{N}$ by induction. □

Exercises:

1. Prove the following propositions fo all positive integers n.

   (a) $1 + 5 + 9 + 13 + \cdots + (4n - 3) = \dfrac{n(4n-2)}{2}$

   (b) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

   (c) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

   (d) $10^1 + 10^2 + 10^3 + \cdots + 10^n = \frac{10}{9}(10^n - 1)$

   (e) $\displaystyle\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$

   (f) $\displaystyle\sum_{k=1}^{n} \frac{1}{(3k-2)(3k-1)} = \frac{n}{3n+1}$ does not work for $n = 1, 2$?

2. Prove the following by induction.

   (a) $2^n \geq 1 + n$ for $n \geq 1$.

   (b) $3^n < (n+1)!$ for $n \geq 4$.

3. Prove that $8^n - 3^n$ is divisible by 5 for all $n \in \mathbb{N}$.

4. Prove that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$.

5. Prove by induction that if p is any real number satisfying $p > -1$, then

$$(1+p)^n \geq 1 + np$$

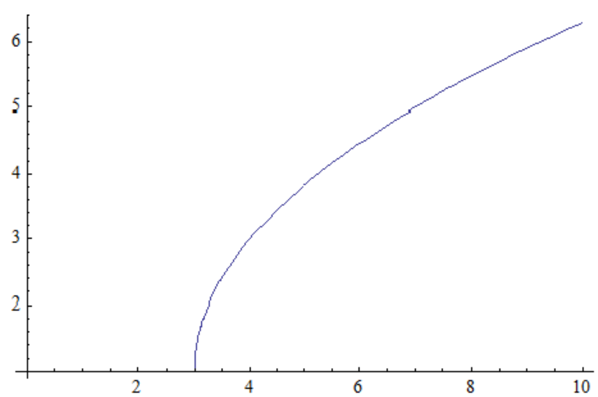
   for all $n \in \mathbb{N}$.

6. Use induction to show that the nth derivative of x^{-1} is $\dfrac{(-1)^n n!}{x^{n+1}}$.

Answers to Test Four
and
Exercises from Worksheets 4.1 - 4.13

Answers to Test Four

1. (a) $\dfrac{3}{3x+2}$ (b) $2x\cos x - x^2\sin x$
2. (a) 16.5 and 21.5 (b) $32/3$
3. (a) $\log x + C$ (b) $\tan x + C$
4. (a) $x = \frac{9}{2}t^2 + 4t + 3$ (b) $1000e$
5. $6x^2 - 5x + 4$
6. (a) 4 (b) $\sqrt{\dfrac{1+\sqrt{2}}{2\sqrt{2}}}$
7. $y = 2\sqrt{x-3} + 1$



8. (a) $\dfrac{\sqrt{x}+1}{\sqrt{x}+2}$ (b) $\dfrac{1}{1-x} - 2$
9. (a) $x \neq \log_e 3$ (b) $\log_e\left(\frac{1}{x} + 3\right)$
10. (a) 15
    (b) $1 + 2^3 + 3^3 + 4^3 + 5^3$
    (c) $\displaystyle\sum_{n=1}^{10} nx^{2n}$

11. (i) 6^3 (ii) $6 \times 5 \times 4$

12. -250

Answers for Worksheet 4.1

Section 1

1. (a) $3\cos 3x$
   (b) $-2\sec^2(-2x)$
   (c) $-12x\sin 6x^2$
   (d) $20(4x+5)^4$
   (e) $18(6x-1)^2$
   (f) $24x(3x^2+1)^3$
   (g) $4e^{4x}$
   (h) $14e^{2x}$
   (i) $\cos x\; e^{\sin x}$
   (j) $-\sin x\; e^{\cos x}$
   (k) $-6(6-2x)^2$
   (l) $-4(7-x)^3$

Section 2

1. (a) $2x\sin x + x^2\cos x$
   (b) $4e^{3x}(1+3x)$
   (c) $xe^{3x}(2+3x)$
   (d) $\cos x - x\sin x$
   (e) $4\log(2x+1) + \frac{8x}{2x+1}$
   (f) $2x\log(x+2) + \frac{x^2}{x+2}$
   (g) $xe^{x^3}(2+3x^3)$
   (h) $(x+1)^2(12x+6)$
   (i) $(x+2)^2(12x+6)$
   (j) $2e^{2x}(4x+1)$
   (k) $e^{2x}(\cos x + 2\sin x)$
   (l) $2e^{4x}(2\cos 2x - \sin 2x)$

Section 3

1. (a) $\dfrac{x^4+3x^2}{(x^2+1)^2}$
   (b) $\dfrac{x^2+2x-3}{(x+1)^2}$
   (c) $\dfrac{5}{(2x+3)^2}$
   (d) $\dfrac{e^{2x}(2x-7)}{(x-3)^2}$
   (e) $\dfrac{x\cos x - 2\sin x}{x^3}$
   (f) $\dfrac{1}{\cos^2 x}$
   (g) $\dfrac{-3x^2-6}{(x^2-2)^2}$
   (h) $\dfrac{-10}{(x-4)^2}$
   (i) $\dfrac{e^x(6x+24)}{(x+5)^2}$
   (j) $\dfrac{e^{2x}(2\sin x - \cos x)}{\sin^2 x}$

Section 4

1. (a) $y = -8x + 2$ (b) $y = -6x + 15$ (c) $y = 71x + 280$

2. (a) $y = \dfrac{x}{24} - \dfrac{241}{6}$ (b) $y = -\dfrac{x}{7} + \dfrac{20}{7}$ (c) $y = \dfrac{2x}{3} - \dfrac{5}{3}$

3. $y = 4x + 1$; $x = -1/4$.

Exercises 4.1

1. (a) $-\dfrac{2}{x^3} - 6$
   (b) $e^{2x}(1 + 2x)$
   (c) $2\cos 2x + 4\sin 4x$
   (d) $(2x + 1)^2(5x + 7)$
   (e) $4(\sin x + x\cos x)$
   (f) $\dfrac{2x}{x^2 + 1}$
   (g) $\log x + 1$
   (h) $2\sin x\cos x$
   (i) $e^x(\sin x + \cos x)$
   (j) $-\dfrac{25}{(x-4)^2}$
   (k) $\dfrac{x^2 + 6x}{(x+3)^2}$
   (l) $\dfrac{e^x(x-3)}{(x-2)^2}$

2. (a) $y = 3ex - 2e^2$
   (b) 0
   (d) Minimum at $(-3/2, -25/4)$.

Answers for Worksheet 4.2

Section 1

1. (a) $\dfrac{2480}{27}$; $\dfrac{1328}{27}$ (b) $\dfrac{161}{2}$; $\dfrac{113}{2}$ (c) $\dfrac{25}{4}$; $\dfrac{17}{4}$

Section 2

1. (a) $2x^3 + 4x^2 - 3x + C$
   (b) $2x^5 - x^3 + 5x + C$
   (c) $\dfrac{3x^5}{5} - 2x^3 - 7x + C$

(d) $\frac{x^2}{2} + 3x + C$

(e) $\frac{x^4}{4} + \frac{1}{2x^2} + x^2 + x + C$

(f) $-\frac{4}{x^2} + \frac{1}{x} + \frac{3x^2}{2} + 4x + C$

(g) $\frac{4x^3}{3} + \frac{7}{3x^3} + 2x + C$

(h) $\frac{x^5}{5} - x^2 + C$

(i) $\frac{63x^6}{6} - x + C$

(j) $-\frac{2}{x^2} + \frac{6}{x} + C$

Section 3

1. (a) 75/2
   (b) 39
   (c) 12
   (d) 76/3
   (e) 33/2
   (f) 4
   (g) 12
   (h) 16
   (i) 38/3
   (j) 16

Exercises 4.2

1. (a) i. The upper limit is $0.2 \times \left(\sqrt{0.2} + \sqrt{0.4} + \sqrt{0.6} + \sqrt{0.8} + 1\right)$.
   The lower limit is $0.2 \times \left(\sqrt{0.2} + \sqrt{0.4} + \sqrt{0.6} + \sqrt{0.8}\right)$.

   ii. The upper limit is $\frac{1}{10} \sum_{k=0}^{9} \frac{1}{1 + \frac{k}{10}}$. The lower limit is $\frac{1}{10} \sum_{k=1}^{10} \frac{1}{1 + \frac{k}{10}}$.

   (b) i. $x + \frac{x^2}{2} + \frac{x^3}{3} + C$ ii. $\frac{2x^{3/2}}{3} + C$ iii. $-\frac{2}{x^2} + C$

   (c) i. 56 ii. 2/3 iii. 7

2. (a) 10
   (b) No
   (c) $\int_1^2 x\,dx$
   (d) 1/3
   (e) 27/4
   (f) 1
   (g) -1/20

3. (a) $-10/3$
   (c) $R = 100q - 0.015q^2 + C$
   (d) $\frac{3x^2}{2} + \frac{2x^3}{3} - \frac{2x^{5/2}}{5}$

Answers for Worksheet 4.3

Exercises 4.3

1. (a)
   i. $-\dfrac{e^{-4x}}{4} + C$
   ii. $2e^{x/2} + C$
   iii. $\log(8 + 7x - 3x^2) + C$
   iv. $\dfrac{\sin 2x}{2}$
   v. $\frac{1}{5}\tan(5x - 2)$
   vi. $-\dfrac{1}{x} - \log x$

   (b)
   i. $\dfrac{1}{2}(e - 1)$
   ii. $\log 3$
   iii. 1
   iv. $\dfrac{1}{3}$

2. (a) $2\log\left(\frac{6}{5}\right)$
   (b) $\dfrac{1}{3}(e^9 - 1)$
   (c) e
   (d) $\sqrt{3} - \dfrac{\pi}{3}$
   (e) $x + 4\log(x + 2) + C$

Answers for Worksheet 4.4

Exercises 4.4

1. (a) $y = \frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x + \frac{89}{12}$
   (b) $y = \frac{2}{3}x^3 + x^2 - \frac{28}{3}$
   (c) $y = \sin x + \frac{7}{2}$
   (d) $y = -(10 - x^2)^{\frac{1}{2}} + 8$

2. (a) 5 (b) 2 (c) $k = \dfrac{\log 2}{10}$ (d) $\dfrac{5}{e}$ (e) $k = \dfrac{\log 2}{20}$

3. (a) $2.9t^2 + 0.2t + 25$
   (b) $\dfrac{50t^3}{3} - 40t^{\frac{3}{2}} + 25000$; $\dfrac{841250}{3}$
   (c) i. 0 ii. 0 iii. $\dfrac{16}{\pi^2} - \dfrac{16}{\pi}$

Answers for Worksheet 4.5

Section 1

1. (a) (i) 1 (ii) 1 (iii) 4
   (b) (i) 9 (ii) 0 (iii) 3
   (c) (i) -2 (ii) 0 (iii) 1
   (d) (i) 6 (ii) -3 (iii) 5
   (e) (i) 0 (ii) 7 (iii) 6
   (f) (i) 7 (ii) -6 (iii) 4

2. $a = 5, \quad b = \pm 2, \quad c = 0, \quad d = 2, \quad e = -7.$

Section 2

1. (a) $-x^2 - 10x, \qquad \deg = 2$
   (b) $4x^5 + 12x^4 - 3x^3 = 10x^2 - 3x, \qquad \deg = 5$
   (c) $x^4 + 4x^3 + 6x^2 + 4x + 1, \qquad \deg = 4$
   (d) $-x^4 - 15x^3 + 4x + 4, \qquad \deg = 4$
2. $p(1) = 4, \qquad p(0) = 4, \qquad p(-2) = 100$
3. (a) $3x^3 - x^2 + 4x + 7 = (x + 2)(3x^2 - 7x + 18) - 29$
   (b) $3x^3 - x^2 + 4x + 7 = (x^2 + 2)(3x - 1) + (-2x + 9)$
   (c) $x^4 - 3x^2 - 2x + 4 = (x - 1)(x^3 + x^2 - 2x - 4)$
   (d) $5x^4 + 30x^3 - 6x^2 + 8x = (x^2 - 3x + 1)(5x^2 + 45x + 124) + (335x - 124)$
   (e) $3x^4 + x = (x^2 + 4x)(3x^2 - 12x + 48) - 191x$
4. (a) $q(x) = x, \qquad r(x) = -x^2 + x - 1$
   (b) $q(x) = x + 3, \qquad r(x) = 3$
   (c) $q(x) = 5x^2 + 7, \qquad r(x) = 2x + 15$
   (d) $q(x) = x^2 - 4x + 11, \qquad r(x) = -29x - 44$
   (e) $q(x) = x^3 - x^2 + x - 1, \qquad r(x) = 2$

Section 3

1. (a) 486 (b) 16

2. (a) 18 (b) 10 (c) 77 (d) 12

Section 4

1. (a) $(x-2)(x^2-x+3)$
   (b) $(x-1)^2(x+5)$
   (c) $x(2x+1)(3x-2)$
   (d) $(4x+1)(x-3)(x+1)$

2. $k=-3$, roots are: 2 (double) and -7

3. $2(x+1)(x-1/3)$

4. $4x^3-20x^2+12x+36$

5. (b) $\frac{1}{3}, -2$

Exercises 4.5

1. (a) $q(x)=x-1, \quad r(x)=15x-12$
   (b) $q(x)=x^2+2x-6, \quad r(x)=33$
   (c) $q(x)=2x^2-1, \quad r(x)=3$
   (d) $q(x)=x^2-2, \quad r(x)=4x-5$

2. (a) $(3x-2)(x+1)(x-3)$
   (b) $(x-2)(x^2-2x+2)$
   (c) $x(2x-1)(x+3)$
   (d) $(x+2)^3$

3. (a) 2, $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$
   (b) $\frac{2}{5}$, -1, -4
   (c) 3, 2
   (d) 1
   (e) 2, -2, -5

4. (a) $a=4$
   (b) Factors are $(x-3)$ and (x^2-x+1)

5. (a) $k=11$, factors are $(x-4)$, $(x+1)$ and $(3x-2)$
   (b) Roots are $4, -1$ and $\frac{2}{3}$

6. $5x^2+14x-3$

7. $x^2 - 3x - 4$

8. $2x^3 - 6x^2 + 2x + 2$

Answers for Worksheet 4.6

Section 1

1. (a) $-\sqrt{3}$ (c) $\dfrac{1}{\sqrt{2}}$ (e) $-\dfrac{2}{\sqrt{3}}$

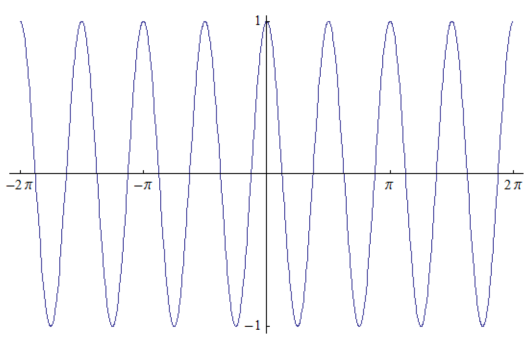
   (b) $\dfrac{\sqrt{3}}{2}$ (d) $-\dfrac{\sqrt{3}}{2}$ (f) 1

2. (a) $\dfrac{5\pi}{6}, \dfrac{7\pi}{6}$ (c) $-\dfrac{\pi}{6}, -\dfrac{2\pi}{3}$ (e) $\dfrac{5\pi}{6}$

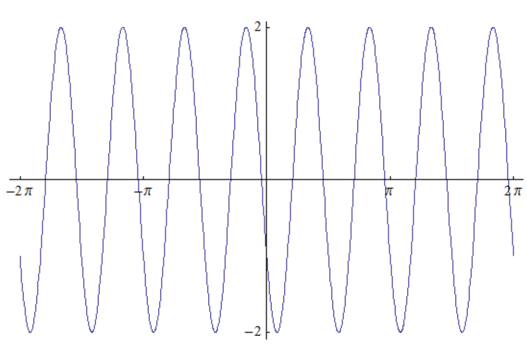
   (b) $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$ (d) $\dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{11\pi}{4}, \dfrac{13\pi}{4}$ (f) $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

Section 2

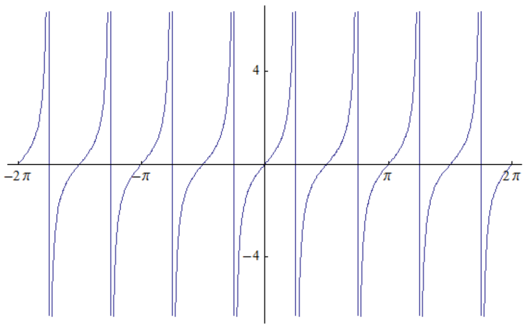
1. (a) period is $\dfrac{\pi}{2}$



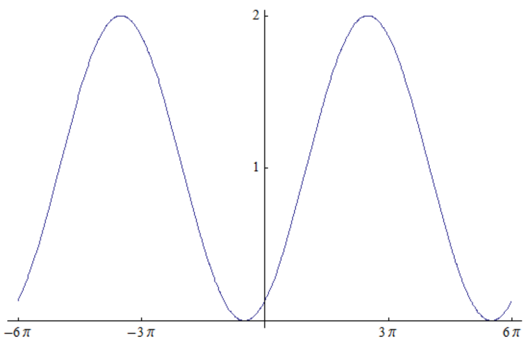
   (b) period is $\dfrac{\pi}{2}$



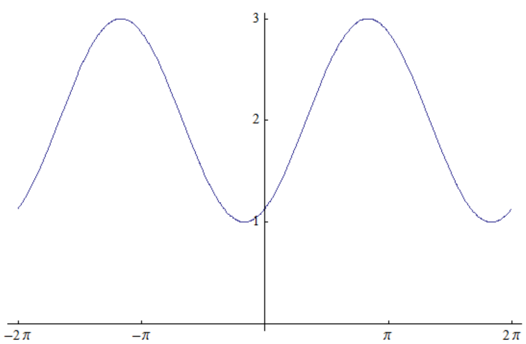
(c) period is $\frac{\pi}{2}$



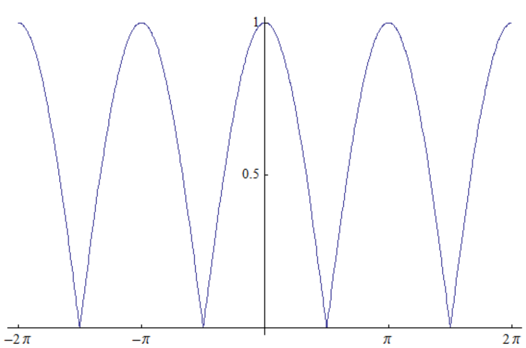
(d) period is 6π



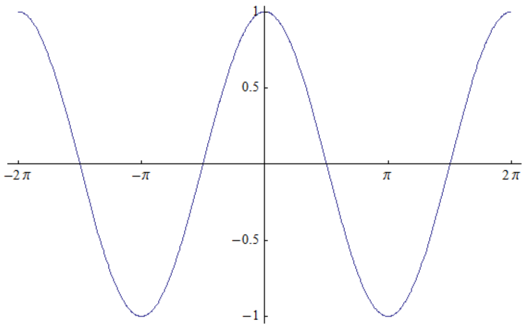
(e) period is 2π



(f) period is π



(g) period is 2π



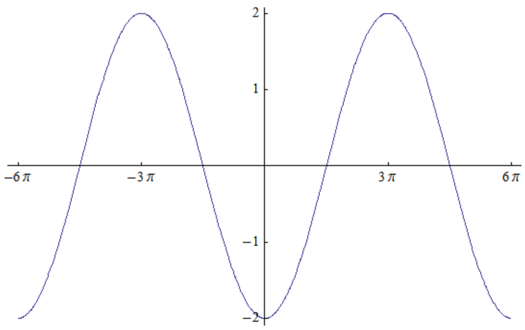
2. (a) $\frac{\pi}{2}, \frac{3\pi}{2}, 0.95, 5.34$
   (b) $\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$
   (c) $0, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
   (d) $\frac{3\pi}{4}, \frac{7\pi}{4}$

Section 3

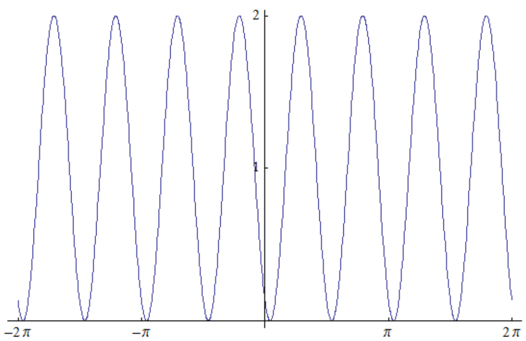
1. (a) $-\sin 4x$
   (b) 1
   (c) $3\cos\left(\frac{x}{4}\right)$
   (d) $\frac{3}{2}$
   (e) $\frac{\sin 4x}{2}$
   (f) 0
2. (a) $\frac{\sqrt{2}-\sqrt{6}}{4}$
   (b) $-\frac{1}{2}$
   (c) $\frac{1}{\sqrt{3}}$
3. (a) $\frac{\sqrt{2+\sqrt{2}}}{2}$
   (b) $\frac{\sqrt{2+\sqrt{3}}}{2}$
   (c) $-\frac{\sqrt{2-\sqrt{3}}}{2}$
4. (a) $\frac{5}{13}$
   (b) $\frac{12}{13}$
   (c) $\frac{120}{169}$
   (d) $\frac{828}{2197}$
   (e) $\frac{276}{715}$

Exercises 4.6

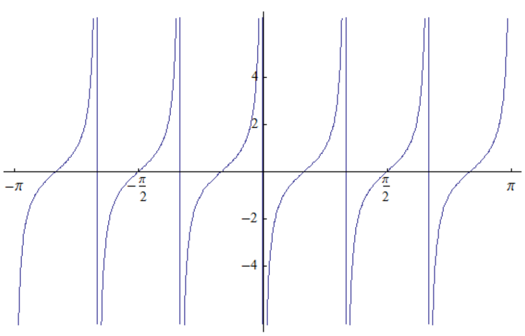
1. (a) $\frac{2\pi}{3}, -\frac{\pi}{3}$
   (b) $\frac{\pi}{12}, \frac{5\pi}{12}$
   (c) $0, \pi, 2\pi, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
   (d) $0, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$
2. (a) $y = -2\cos(\frac{x}{3})$



(b) $y = 1 + \sin(x + \frac{\pi}{3})$



(c) $y = \tan\left(2(x - \frac{\pi}{6})\right)$



3. Proofs only.

4. (a) $-\dfrac{7}{25}$ (b) $-\dfrac{24}{75}$ (c) $\dfrac{7}{24}$

5. (a) $-\dfrac{119}{54}$ (b) $-\dfrac{61\sqrt{5}}{72}$

6. $-\sqrt{2+\sqrt{2}}2$

7. (a) $\sqrt{\dfrac{4+\sqrt{2}+\sqrt{6}}{8}}$ (b) $-\dfrac{1}{2}$

Answers for Worksheet 4.7

Section 1

1. (a) Domain is $(0,5)$, range is $(-2,23)$, codomain is $\mathbb{R}$.
   (b) Domain is $[-1,7]$, range is $[0,36]$, codomain is $\mathbb{R}$.
2. A suitable domain is $[1,\infty)$ and a suitable codomain is $[0,\infty)$.
3. (a) $\{x \in \mathbb{R} : x \geq 0\}$
   (b) $\{x \in \mathbb{R} : x > 0\}$
   (c) $\{x \in \mathbb{R} : x \neq \frac{2}{3}\}$
   (d) $\mathbb{R}$
   (e) $\{x \in \mathbb{R} : x \geq 7\}$
   (f) $\{x \in \mathbb{R} : x \leq -4 \quad \text{or} \quad x \geq 4\}$
4. (a) Domain is $\mathbb{R}$. Range is $[1,\infty)$.
   (b) Domain is $[-9,\infty)$. Range is $[0,\infty)$.
   (c) Domain is $\mathbb{R}$. Range is $\mathbb{R}$.

Answers for Worksheet 4.8

Section 1

1. (a) 0
   (b) $\sqrt{\dfrac{7}{2}}$
   (c) $1-x$
   (d) $\sqrt{\dfrac{1}{2x}+3}$
   (e) $\dfrac{1}{2\sqrt{x+3}}$
   (f) $1-x^2$
   (g) $1-\dfrac{1}{2x}$
2. The number -4 is not in the domain of g.
3. We have $(s \circ t)(x) = \sqrt{x^2+2x+1}$. Its domain is $\mathbb{R}$ and its range is $[0,\infty)$.
4. We have $f(x) = (g \circ h)(x)$ where $g(x) = \sqrt{x}$ and $h(x) = \dfrac{1}{x^2+2}$. Another possibility is $g(x) = \sqrt{\dfrac{1}{x+2}}$ and $h(x) = x^2$.

Section 2

2. (a) and (e); (b) and (f); (c) and (d).

Section 3

1. (a) $y = \dfrac{x}{3}$
   (b) $y = x + 7$
   (c) $y = 2(x - 17)$
   (d) $y = -\dfrac{4}{5}x + \frac{1}{5}$
   (e) $y = \dfrac{3}{x} - 8$
   (f) $y = x^{\frac{1}{3}}$

2. (a) $y = (x - 2)^2$ with domain $[2, \infty)$ and range $[0, \infty)$.
   (b) $y = x^2 - 2$ with domain $[0, \infty)$ and range $[-2, \infty)$.

3. There are two possibilities.
   (a) Restrict the domain of $y = x^2 - 2$ to $[0, \infty)$. The inverse is $y = \sqrt{x + 2}$ with domain $[-2, \infty)$ and range $[0, \infty)$.
   (b) Restrict the domain of $y = x^2 - 2$ to $(-\infty, 0]$. The inverse is $y = -\sqrt{x + 2}$ with domain $[-2, \infty)$ and range $(-\infty, 0]$.

Exercises 4.8

1. The compositions and their domains are:

	Function	Equation	Domain
(a)	$(f \circ g)$	$2x^2 - 6x - 5$	$\mathbb{R}$
	$(g \circ f)$	$4x^2 - 26x + 40$	$\mathbb{R}$
(b)	$(f \circ g)$	$\sqrt{2x^2 - 1}$	$(-\infty, -\frac{1}{\sqrt{3}}] \cup [\frac{1}{\sqrt{3}}, \infty)$
	$(g \circ f)$	$3x - 1$	$[\frac{1}{3}, \infty)$
(c)	$(f \circ g)$	$\sqrt{\frac{3}{x} - 1}$	$(-\infty, 0) \cup (0, 3)$
	$(g \circ f)$	$\frac{1}{\sqrt{3x - 1}}$	$(\frac{1}{3}, \infty)$
(d)	$(f \circ g)$	$\sqrt{\frac{3x^2 - x + 3}{x}}$	$(0, \infty)$
	$(g \circ f)$	$\frac{3x}{\sqrt{3x - 1}}$	$(\frac{1}{3}, \infty)$
(e)	$(f \circ g)$	$x^2 - 3$	$\mathbb{R}$
	$(g \circ f)$	$\lvert x^2 - 3 \rvert$	$\mathbb{R}$

2. Some possibilities are:
   (a) $(f \circ g)(x) = \sqrt{x^3 - 1}$ where $f(x) = \sqrt{x}$ and $g(x) = x^3 - 1$,
   (b) $(f \circ g)(x) = (3x - 4)^3$ where $f(x) = x^3$ and $g(x) = 3x - 4$,
   (c) $(f \circ g)(x) = \dfrac{1}{x^2 - 1}$ where $f(x) = \dfrac{1}{x}$ and $g(x) = x^2 - 1$,
   (d) $(f \circ g)(x) = x$ where $f(x) = x^2$ and $g(x) = \sqrt{x}$.

3. (a) $\dfrac{x}{3} - \dfrac{2}{3}$ (b) $-\dfrac{1}{x} + 4$ (c) $\dfrac{3}{1 - x} - 5$ (d) $(x - 1)^{1/3}$

4. (a) $f^{-1}(x) = \sqrt{x - 2} + 1$ with domain $[2, \infty)$.
   (b) $h^{-1}(x) = -x$ with domain $(0, \infty)$.
   (c) $g^{-1}(x) = \sqrt{9 - x}$ with domain $(-\infty, 9]$.

Answers for Worksheet 4.9

Section 1

1. (a) $486x^{\frac{23}{2}}y^{-\frac{17}{14}}$
   (b) 3^{5x}
   (c) $-1/4$
   (d) $-3/2$
   (e) $\log x + \log y$
   (f) 2

2. The domain of $y = \log_5 x$ is $(0, \infty)$ and its range is $\mathbb{R}$. Its inverse is $y = 5^x$ with domain $\mathbb{R}$ and range $(0, \infty)$.

Section 2

2. (a) $(-3, \infty)$
   (b) $(-3, \infty)$
   (c) $(-3, \infty)$
   (d) $\{x \in \mathbb{R} \mid x \neq 3\}$

3. (a) $(5, \infty)$
   (b) $(5, \infty)$
   (c) $\{x \in \mathbb{R} \mid x \neq 5\}$
   (d) $(-\infty, -2) \cup (2, \infty)$

Section 3

1. (a) $f^{-1}(x) = \log x + 2$

2. (a) $\dfrac{1}{\log x}$
   (b) $3 - e^x$
   (c) $e^{2-x} - 1$

3. (a) $\frac{1}{2}\log x$; $[e^2, \infty)$
   (b) $\dfrac{e^x}{2} - 2$; $\mathbb{R}$

4. One reason is that $\log(-1)$ is not defined. Another reason is that the function e^x is always positive valued.

Exercises 4.9

1. The domain and range are, respectively,

   (a) $(-\frac{3}{2}, \infty)$ and $\mathbb{R}$

   (b) $(0, \infty)$ and $\mathbb{R}$

   (c) $\mathbb{R}$ and $(0, \infty)$

   (d) $[\frac{1}{2}, \infty)$ and $[\log 3, \infty)$

4. (a) $\log x + 2$

   (b) $\log(x+5) - 4$

   (c) e^{x+2}

   (d) $-e^x$

   (e) $2^{x-3} - 4$

   (f) $1 - \log_2 x$

5. (a) $k(x) = \log(x-2)$

   (b) $(2, \infty)$

   (c) $\mathbb{R}$

   (d) $k^{-1}(x) = e^x + 2$

6. There are two possibilities.

   (a) If we restrict the domain of f to $(-3, \infty)$ then its inverse will be $f^{-1}(x) = e^x - 3$.

   (b) If we restrict the domain of f to $(-\infty, -3)$ then its inverse will be $f^{-1}(x) = -e^x - 3$.

Answers for Worksheet 4.10

Section 1

1. 362880

2. 10

Section 2

1. (a) $1^4 + 2^4 + 3^4 + 5^4 + 6^4$

   (b) $\frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7}$

   (c) $3 + 5 + 7 + 9 + \cdots + (2n-1)$

   (d) $2 + 4x + 8x^2 + 16x^3 + \cdots + 2^{n+1}x^n$

   (e) $1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \cdots + \frac{(-1)^n x^n}{2n+1}$

2. (a) $\sum_{n=1}^{6} n^2$

   (b) $\sum_{n=2}^{8} (-1)^n (2n-1)$

   (c) $\sum_{n=0}^{5} \frac{1}{3n+2}$

   (d) $\sum_{k=1}^{n} \frac{k+1}{k+2}$

   (e) $\sum_{k=1}^{2n+1} (-1)^{k+1} 2^k$

   (f) $\sum_{n=1}^{15} 2nx^{2n+1}$

(g) $\sum_{n=0}^{5}(-1)^n \frac{x^{n+1}}{n!}$

(h) $\sum_{n=0}^{5}(4n+3)x^{n+1}$

(i) $\sum_{k=3}^{n}(2k+2)x^{k+1}$

(j) $\sum_{k=4}^{n}k(k-1)x^k$

(k) $\sum_{n=1}^{99}\frac{(-1)^{n+1}x^{2n+1}}{n+1}$

Section 3

1. (a) $x^2 : \frac{3}{2}, \quad x^6 : \frac{7}{6!}$
   (b) $x^2 : 20, \quad x^6 : 72$
   (c) $x^2 : \frac{1}{3}, \quad x^6 : -\frac{1}{4}$
   (d) $x^2 : 13, \quad x^6 : 33$
   (e) $x^2 : 0, \quad x^6 : \frac{1}{5!} - \frac{1}{4!}$
   (f) $x^2 : 4, \quad x^6 : 20$

Exercises 4.10

1. (a) $-x + 4x^2 - 9x^3 + 16x^4 - 25x^5 + 36x^6 - 49x^7$
   (b) $\frac{2}{4}x^6 + \frac{3}{5}x^8 + \frac{4}{6}x^{10} + \frac{5}{7}x^{12} + \frac{6}{8}x^{14} + \frac{7}{9}x^{16}$
   (c) $2x^6 + 6x^9 + 12x^{12} + 20x^{15} + \cdots + n(n+1)x^{3n+3}$

2. (a) $\sum_{n=1}^{6}\frac{(-1)^{n+1}}{n^2}$
   (b) $\sum_{n=0}^{8}(-1)^n(3n+7)$
   (c) $\sum_{k=2}^{n}(k+2)^2$
   (d) $\sum_{k=2}^{n}\frac{1}{(k+5)!}$
   (e) $\sum_{n=1}^{12}3nx^{2n}$
   (f) $\sum_{n=0}^{12}\frac{x^{2n+7}}{n!}$
   (g) $\sum_{k=1}^{10}(-1)^{k+1}(4k+1)x^k$
   (h) $\sum_{k=5}^{n}\frac{kx^{k-2}}{2}$
   (i) $\sum_{k=6}^{n+1}nx^{2n}$

3. (a) $x : -\frac{1}{11!}, \quad x^3 : -\frac{1}{19!}, \quad x^7 : -\frac{1}{35!}$
   (b) $x : \frac{3}{25}, \quad x^3 : \frac{27}{29}, \quad x^7 : \frac{3^7}{121}$
   (c) $x : -1, \quad x^3 : \frac{4}{3}, \quad x^7 : \frac{31}{14}$
   (d) $x : -\frac{1}{3}, \quad x^3 : -\frac{1}{2}, \quad x^7 : -\frac{1}{3}\left(\frac{1}{6!} + \frac{1}{5!}\right)$
   (e) $x : \frac{15}{7!} + \frac{1}{3!}, \quad x^3 : \frac{25}{11!} + \frac{3}{7!}, \quad x^7 : \frac{45}{19!} + \frac{7}{15!}$

4. (a) $(-(n+1)^2$
   (b) $-\frac{n}{n+1}$

Answers for Worksheet 4.11

Section 1

1. 24
2. $26^3 \times 10^3$
3. 2652

Section 2

1. 5040, 210
2. 2520, 21
3. 43758, 73440
4. $^{12}C_3$
5. $^{12}P_3$

Section 3

1. (a) 256 (b) 24
2. 504
3. (a) 5040 (b) 240 (c) 1440 (d) 3600
4. (a) 360 (b) 20160 (c) 37800

Section 4

1. $^6C_2$
2. $^{12}C_3$
3. (a) $^{13}C_6$
   (b) $^{13}C_3 \times ^{12}C_3$
   (c) $^{23}C_4$

Exercises 4.11

1. (a) Not important, $^6C_3$
   (b) Not important, $^8C_5$
   (c) Important, $^8P_3$
   (d) Important (if stamps are distinct), $^9P_5$
2. $^{13}C_1 \times ^4C_3 \times ^{12}C_1 \times ^4C_2$
3. (a) $^{52}C_{12}$ (b) $^4C_1 \times ^{13}C_{12}$ (c) $^4C_3 \times ^4C_3 \times ^{44}C_6$ (d) $^{13}C_{12} \times 4^{12}$

4. (a) ${}^{11}P_{10}$ (b) $10!$

5. ${}^{30}C_3, \quad {}^{30}P_3$

6. (a) $5!$ (b) $5!$ (c) $3! \times 4!$

7. ${}^{30}C_5$

8. (a) 10^4 (b) ${}^{10}P_4$

9. (a) $\frac{7!}{2!2!}$ (b) $\frac{6!}{2!2!}$ (c) $\frac{6!}{2!}$ (d) $2! \times 2!$

10. (a) $\frac{7!}{7}$ (b) $2! \times \frac{6!}{6}$ (c) $3! \times \frac{5!}{5}$

11. (a) ${}^{10}C_5$ (b) ${}^{6}C_5$ (c) 246 (d) 186

12. (a) ${}^{13}C_5$ (c) ${}^{13}C_3 \times {}^{13}C_2$
    (b) ${}^{4}C_1 \times {}^{13}C_5$ (d) ${}^{4}C_1 \times {}^{13}C_3 \times {}^{3}C_1 \times {}^{13}C_2$

13. (a) ${}^{6}C_3$ (b) ${}^{5}C_3 + {}^{6}C_3 + {}^{4}C_3$ (c) $5 \times 6 \times 4$

14. (a) $10!$ (b) ${}^{5}P_2 \times 8!$ (c) $2 \times 5 \times 5 \times 8!$

15. $2! \times {}^{28}\mathrm{C}_8 \times 9!$

16. ${}^{12}C_3 \times 3!$

Answers for Worksheet 4.12

Section 1

1. Not given
2. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
3. 11^{th} line
4. $1, 8, 28, 56, 70, 56, 28, 8, 1$ and $1, 9, 36, 84, 126, 126, 84, 36, 9, 1$
5. $32a^5 - 240a^4 + 720a^3 - 1080a^2 + 810a - 243$

Section 2

1. (a) $1 - 4x^2 + 6x^4 - 4x^6 + x^8$
   (b) $x^9 - 3x^4 + \frac{3}{x} - \frac{1}{x^6}$
2. ${}^5C_1 \times 2^4, \qquad {}^5C_2 \times 2^3, \qquad {}^5C_3 \times 2^2$
3. ${}^7C_1 \times (-2)^6, \qquad {}^7C_2 \times (-2)^5, \qquad {}^7C_4 \times (-2)^3$
4. ${}^7C_2 \times 2^2(-3)^5, \qquad 0$
5. 0.729

Exercises 4.12

1. (a) i. $16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$
   ii. $a^6 + 6a^4 + 15a^2 + 20 + \frac{15}{a^2} + \frac{6}{a^4} + \frac{1}{a^6}$
   iii. $\dfrac{a^7}{b^7} - \dfrac{7a^5}{b^5} + \dfrac{21a^3}{b^3} - \dfrac{35a}{b} + \dfrac{35b}{a} - \dfrac{21b^3}{a^3} + \dfrac{7b^5}{a^5} - \dfrac{b^7}{a^7}$
   iv. $x^{10} - 10x^8 + 40x^6 - 80x^4 + 80x^2 - 32$
2. (a) i. 56
   ii. 378
   (b) i. -5376
   ii. 8064
3. 1.4641