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Test Three

This is a self-diagnostic test. Every pair of questions relates to a worksheet in a series available in the MUMS the WORD series. For example question 5 relates to worksheet 3.5 *Simultaneous Equations*. If you score 100% on this test and test 4 then we feel you are adequately prepared for your first year mathematics course. For those of you who had trouble with a few of the questions, we recommend working through the apppropriate worksheets and associated computer aided learning packages in this series.

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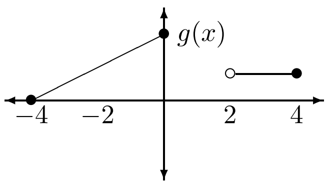
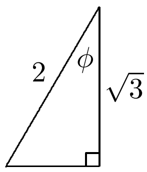
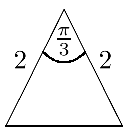
1. (a) If $f(x) = x^2 + 2$, what is $f(3x)$?
   (b) If $g(x) = x^2$, and $f(x) = 2x + 1$, what is $g \circ f(x)$?
2. (a) What is the domain of $g(x)$ in the following graph?



   (b) At what points on the graph above does $g(x) = 0$?
3. (a) If an angle is 60°, how many radians is it?
   (b) For the △ drawn below, what is the angle ϕ?



4. (a) What is $\sin \frac{7\pi}{4}$? (Without a calculator)
   (b) Find the area of the triangle drawn below.



5. (a) Given:

$$\begin{aligned} y &= 2x + 5 \\ y &= kx + 4 \end{aligned}$$

For what value(s) of k will the system have no solutions?

(b) Solve the system:

$$\begin{aligned} y &= 3u + 6 \\ 3y &= 5u + 2 \end{aligned}$$

6. (a) If $2, 7$ are the first two terms of an arithmetic progression, what is the 10th term?

(b) What is the sum of the following infinite geometric series?

$$1 + \frac{9}{10} + \frac{81}{100} + \ldots$$

7. (a) Find the limit of

$$\lim_{n \to 5} \frac{x^2 - 25}{x - 5}$$

(b) Is the function

$$f(x) = \begin{cases} \frac{x^2+6}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

continuous at $x = 0$? Why?

8. (a) Find the derivative of $f(x) = x^3 + 3x^2 + 3$.

(b) What are the stationary points of the function $g(x) = x^3 + 3x^2$?

9. (a) What sort of turning points does the function $f(x) = 6 - 3x^2$ have?

(b) When does the concavity change for the function $h(x) = x^3 + x^2 + 5x + 2$?

10. (a) Differentiate $y = \sin(5x + 3)$.

(b) Differentiate $y = 3e^x$.

Worksheet 3.1 Functions

Section 1 Definitions

What is a function? A function can be thought of as a machine. It accepts an input, applies a rule to it and then produces an output. Diagrammatically, we might view the process like:

$$\text{input} \quad \rightarrow \quad \text{rule} \quad \rightarrow \quad \text{output}$$

Example 1 : Rule f: take the input, and multiply it by 5. If we apply rule f to the input 4, we get $5 \times 4 = 20$.

$$4 \quad \rightarrow \quad 5 \times 4 \quad \rightarrow \quad 20$$

What is the output when we apply rule f to the input x?

$$x \quad \rightarrow \quad 5 \times x \quad \rightarrow \quad 5x$$

As mentioned in other worksheets we look for shorthand ways of working with things. The shorthand way of writing "apply rule f to input 4" is to write $f(4)$. We say this as f of 4. So

$$f(4) = 20 \quad \text{and} \quad f(x) = 5x$$

We say the second item as f of x. When we apply rule f to input x our output gives us a shorthand way of writing the actual rule.

Example 2 : We define rule G: take the input squared, and then add 5. Apply rule G to the inputs $-1, 1, a+1$ and x.

$$\begin{aligned} G(-1) &= (-1)^2 + 5 = 6 \\ G(1) &= (1)^2 + 5 = 6 \\ G(a+1) &= (a+1)^2 + 5 = a^2 + 2a + 6 \\ G(x) &= x^2 + 1 \end{aligned}$$

There are several different ways of representing functions. The most common ways are

1. As a table of values
2. As a graph
3. As an algebraic expression

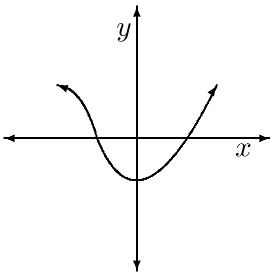
Here are some examples of the ways of representing a function.

Example 3 :

input	0	1	2	3	4	5	6
output	5	6	7	8	9	10	11

This is a table of values, and the rule isn't given explicitly in this case. However, we might be able to guess an appropriate rule. In this case it appears to be to take the input and add 5. With a table of values, the rule will not usually be given, and it may not be obvious what the rule is. But the table of values still represents a function. Let's give the rule that converts an input into an output a name, say f. Then the function that is associated with this table is $f(x) = x + 5$.

Example 4 :



This graph also represents a function, and from the graph we can learn things about the function. We will talk more about graphs in the next worksheet.

Example 5 : Consider

$$f(x) = 5x^2 + 2$$

Here we use the shorthand notation for the function rule, and the input is always some variable which is often x or t. The output is given by what the rule does to the variable, in this case x. Rule f in this case says take the input, square it, multiply the result by 5, and then add 2. This is the output.

Exercises:

1. For each part, find a function which describes the table of values

(a)

x	0	1	2	3
$f(x)$	-4	-3	-2	-1

(b)

x	2	4	5	6
$f(x)$	5	17	26	37

(c)

x	0	2	4	5
$f(x)$	1	5	9	11

2. Complete the table of values for the function $f(x) = 3 - x^2$.

x	-3	-1	2	3	5	7
$f(x)$						

Section 2 Substitution

When a function is represented algebraically, we are given the rule as it applies to some variable. This is called functional notation. To compute the rule applied to any input we simply replace the variable with the input.

Example 1 :

$$\begin{aligned} \text{Given:}\quad f(x) &= x^2 \text{ then} \\ f(5) &= (5)^2 = 25 \\ f(-1) &= (-1)^2 = 1 \\ f(a+b) &= (a+b)^2 = a^2 + 2ab + b^2 \\ f(2y) &= (2y)^2 = 4y^2 \end{aligned}$$

Example 2 :

$$\begin{aligned} \text{Given:}\quad g(t) &= 5t - 3 \text{ then} \\ g(1) &= 2 \\ g(0) &= -3 \\ g(10) &= 47 \\ g(x^2) &= 5x^2 - 3 \end{aligned}$$

Example 3 :

$$\begin{aligned} \text{Given: } h(x) &= \frac{1}{x} \text{ then} \\ h(1) &= \frac{1}{1} = 1 \\ h(3) &= \frac{1}{3} \end{aligned}$$

In example 3, $h(x) = \frac{1}{x}$. So $h(0)$ doesn't make sense since we can't divide by zero. When the function doesn't make sense for a particular input value, we say that the function is not defined for that input value.

Example 4 :

$$\begin{aligned} \text{Given: } f(x) &= 3x^2 + 2x + 2 \text{ then} \\ f(1) &= 3 \times (1)^2 + 2 \times 1 + 2 = 7 \\ f(0) &= 2 \\ f(-1) &= 3 \\ f(3y) &= 27y^2 + 6y + 2 \end{aligned}$$

Exercises:

1. Given $f(x) = 2x - x^2$, find
   (a) $f(3)$
   (b) $f(-2)$
   (c) $f(x-1)$
2. Given $f(x) = 2x^2 - x + 3$, find
   (a) $f(4)$
   (b) $f(-3)$
   (c) $f(x+2)$
3. Given $f(x) = (x+2)^2 - x + 3$, find
   (a) $f(2)$
   (b) $f(-2)$
   (c) $f(-4)$
   (d) $f(x+1)$

Section 3 Composition of Functions

This section deals with a thing called composition of functions. As a picture, composition looks like this:

$$\text{input}_1 \;\rightarrow\; \text{rule}_1 \;\rightarrow\; \text{output}_1 = \text{input}_2 \;\rightarrow\; \text{rule}_2 \;\rightarrow\; \text{output}_2$$

It is like having two machines, one after the other. The result from one machine forms the input to the next machine. We could also write it like this

$$x \;\rightarrow\; \text{rule } f \;\rightarrow\; f(x) \;\rightarrow\; \text{rule } g \;\rightarrow\; g(f(x))$$

So $g(f(x))$ is a composite function which may also be written $g \circ f(x)$. The circle may be taken to mean 'follows'.

Note: It is important to realize that

$$g \circ f(x) \neq f \circ g(x)$$

The order in which the functions are applied is important. It is equally, if not more important, to realize that

$$f \circ g(x) \neq f(x) \times g(x)$$

Example 1 : Take $g(x) = x^2$ and $f(x) = 1 + x$. Then

$$\begin{aligned} g \circ f(1) &= g(f(1)) = g(2) = 4 \\ g \circ f(3) &= g(f(3)) = g(4) = 16 \\ f \circ g(1) &= f(g(1)) = f(1) = 2 \\ f \circ g(3) &= f(g(3)) = f(9) = 10 \\ g \circ f(x) &= g(f(x)) = g(x+1) = (x+1)^2 \\ f \circ g(x) &= f(g(x)) = f(x^2) = x^2 + 1 \end{aligned}$$

When dealing with compositions, if you write it all out longhand as in the above examples, you shouldn't get too confused. It's when you try and do too much in your head that you get the computation around the wrong way.

Exercises:

1. Given $f(x) = x + 2$ and $g(x) = 2x$, find
   (a) $f(3)$ (c) $g \circ f(3)$ (e) $f \circ g(3)$
   (b) $g(5)$ (d) $g \circ f(-1)$ (f) $f \circ g(-4)$
2. Given $f(x) = x^2 - 1$ and $g(x) = 3 - x$, find
   (a) $g \circ f(1)$ (b) $g \circ f(t)$ (c) $g \circ f(4)$ (d) $f \circ g(x+1)$ (e) $f \circ g(x+2)$

Section 4 Functions from words

Functions are useful for determining the answer to many problems that occur in real-life situations. You may be required to take a problem that is given to you in words and come up with the function describing the information given. There is no hard and fast method of dealing with problems, although there are some general hints.

1. Read the information carefully, and translate as much as possible into mathematical expressions.
2. Try out a few inputs to get a feel for the rule before writing it down with a variable.

Example 1 : You have to create a rectangular paddock and you only have 1000 metres of fencing. You can choose the width of the paddock. Find a function that takes as input the width of the paddock and gives as an output the area of the paddock enclosed.

The perimeter of the paddock is 1000m. If the length is l and the width w then the perimeter p in symbols is $p = 2l + 2w$. The area A is $A = lw$, which we will try to write in terms of w, rather than w and l. From $2l + 2w = 1000$, we see that

$$\begin{aligned} 2l &= 1000 - 2w \\ l &= \frac{1}{2}(1000 - 2w) \\ &= 500 - w \end{aligned}$$

Then the area is

$$\begin{aligned} A &= lw \\ &= (500 - w)w \\ &= 500w - w^2 \end{aligned}$$

We can create the following table:

w	l	A
300	$500 - 300 = 200$	300×200
200	$500 - 200 = 300$	200×300
100	$500 - 100 = 400$	100×400
x	$500 - x$	$x(500 - x)$

The required function is

$$A = f(x) = x(500 - x)$$

where A is an obvious symbol to represent area.

In later worksheets on differentiation we will learn a technique that will allow us to find the optimum width so that fencing of the paddock will yield the maximum area.

Example 2 : A ferry carries an average of 300 people a day. The fare is \$ 1.20. The UTA research shows that 50 extra people will travel per day for every 10cent fare reduction. Work out the function that has the number of fare reductions as input, and as output the total amount of money collected by the UTA each day.

Reductions	Fare	Number of People	Money collected
0	1.20	300	300×1.20
2	1.00	$300 + 2 \times 50$	$(300 + 2 \times 50) \times 1.00$
12	0.00	$300 + 12 \times 50$	$(300 + 12 \times 50) \times 0.00$
x	$1.20 - 0.10x$	$300 + x \times 50$	$(300 + x \times 50) \times (1.20 - 0.10x)$

So the function in terms of the number of reductions is

$$(300 + x \times 50) \times (1.20 - 0.10x)$$

In the worksheet on differentiation we will learn how to maximize the money taken in by the UTA.

Exercises:

1. A truck weighs 1500 kg and it is to be loaded with cartons each weighing 5 kg. Work out the function which has the number of cartons as input and the total weight of the truck as output.

2. A photocopier service costs \$240 plus 2.5 cents for every copy made. Work out the function which has the number of copies made as the input and the total cost as output.

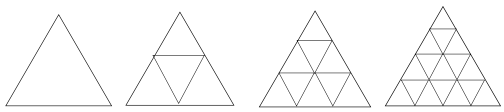
Exercises for Worksheet 3.1

1. (a) A mother records the height of her son over the first 8 months, and measurements she made are shown in this table:

Input (age in months)	0	2	4	6	8
Output (height in cm)	50	54	58	62	66

   Express the output as a function of the input.

   (b) Consider the pattern of triangles shown:



   If the input is the number of horizontal rows in the pyramid, and the output is the number of triangles, describe the relationship between the input and the output.

   (c) Evaluate x, given that $x = 2a^3 - 3\sqrt{a}$, when $a = 2.73$.

   (d) Evaluate a, given that $v = \sqrt{u^2 + 2as}$, when $v = 10$ m/s, $u = 9$ m/s, and $s = 2$ m.

   (e) The formula for converting degrees Fahrenheit (F) to Celsius (C) is given by $C = \frac{5}{9}(F - 32)$. Evaluate C when $F = 100$.

   (f) If $f(x) = 3x^2 + 1$, find $f(-2)$.

   (g) If $g(x) = \frac{3x+2}{x^2-1}$, find $g(\frac{1}{2})$.

   (h) If $f(x) = x^2 - 2x + 3$, find $f(x + h)$.

   (i) If $f(x) = 3x^2 - 2x + 4$, find $\frac{f(x+h)-f(x)}{h}$.

   (j) For $f(x)$ given in the previous question, evaluate $\frac{f(x+h)-f(x)}{h}$ when $x = 2$ and $h = 0.001$.

2. (a) If $f(x) = x^2$ and $g(x) = \frac{1}{x}$, find

   i. $f \circ g(x)$

   ii. $g \circ f(x)$.

   (b) If $f(x) = 3x^2$ and $g(x) = x - 3$, find

   i. $f \circ g(x)$

   ii. $g \circ f(x)$.

   (c) If $f(x) = x^2$, $g(x) = x + 1$, and $h(x) = 2x$, find $f \circ g \circ h(x)$.

   (d) If $f(x) = x^2$, find

   i. $f(2)$

   ii. $f(x + h)$

   iii. $f(2x)$

iv. $f(x+1)$.

(e) If $f(x) = \frac{1}{2x+1}$, find

i. $f(\frac{1}{2})$

ii. $f(3+x)$

iii. $f(x^2)$.

3. (a) Here is a rule for a function: take the input, multiply it by 3, then add 4, then square the result.

   i. Express the output as a function of the input.

   ii. Evaluate the output when the input equals -2.

   (b) Here is another rule for a function: take the input, subtract 2, take the square root of the result, then add 5.

   i. Express the output as a function of the input.

   ii. Evaluate the output when the input equals 2.

   (c) There are 27 times as many cars as motorcycles in Australia. If C represents the number of cars, and M the number of motorcycles, write an equation describing the relationship between M and C.

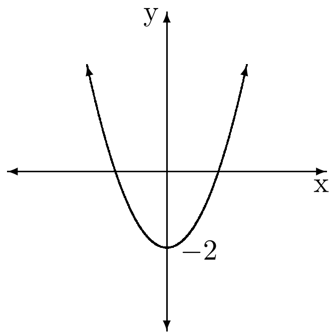
Worksheet 3.2 Graphs

Section 1 Range & Domain

In the last worksheet we mentioned that functions can be represented as graphs. Graphs have already been referred to in worksheet 2.10 when we looked at graphing a straight line. The graph of a function is the collection of all points (x, y) that satisfy a given function. From looking at graphs we can learn a lot of information about the function it represents. From a graph, we can make estimates about the value of the function at certain inputs; we can see where maximum and minimum values of the function are; we can see how rapidly the function is increasing or decreasing.

Two important pieces of information we can read off a graph are the range and domain of the function. The range of a function is all the values that the function takes. So if y is a function of x, the range is all the y-values that can be taken. The range may be written in one of several ways - typically as an interval, or using inequality signs.

Example 1 :



The arrows on the graph of $f(x)$ indicate that it keeps going upwards. The range of $f(x)$ can be written as

$$-2 \leq y$$

That is, y can be greater than or equal to -2.

In interval notation this is written as $[-2, \infty)$.

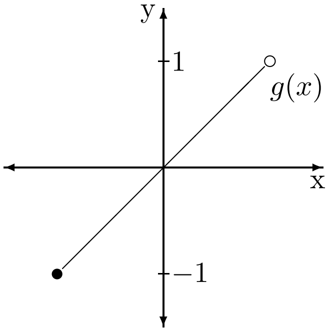
Note: The square bracket indicates that -2 is included in the range. The round bracket indicates all numbers up to but not including the end part. So

$$(2, \infty) \quad \text{is the same as} \quad y > 2$$

and

$$[1, 3) \quad \text{is the same as} \quad 1 \leq y < 3$$

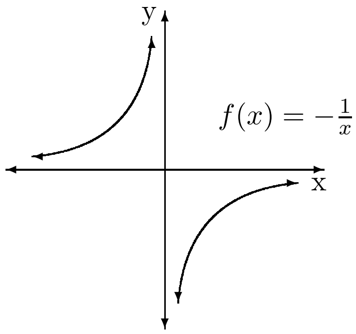
Example 2 :



The open circle on a graph means the same as an open circle on a number line: all numbers up to but not including the point. Thus the range of $g(x)$ is $-1 \leq y < 1$, or in interval notation $[-1, 1)$.

The domain of a function is all the inputs that make sense. In other words, for a function $f(x)$ it is all the x-values for which the function is defined. The domain of a function is normally written in the same notation as the range i.e. either using inequalities or interval notation.

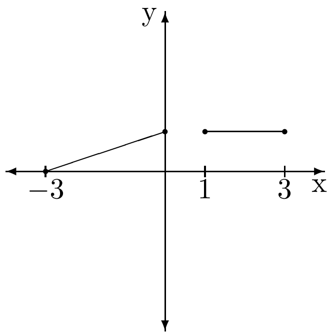
Example 3 :



The domain is $x > 0$ and $x < 0$. Or equivalently, $(-\infty, 0)$ and $(0, \infty)$. $f(x)$ does not make sense for $x = 0$, so it is not included in the domain.

The range of this function is $y > 0$ and $y < 0$. Or, equivalently, $(-\infty, 0) \cup (0, \infty)$.

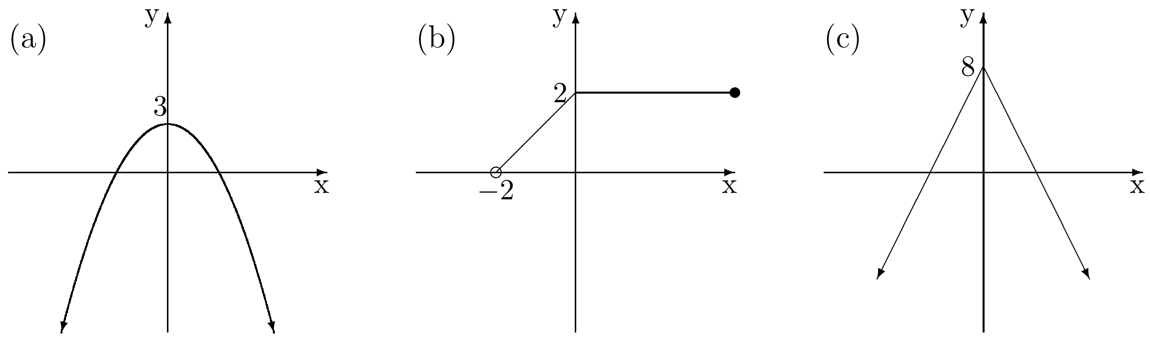
Example 4 :



The domain of this graph is $-3 \leq x \leq 0$ and $1 \leq x \leq 3$, or in interval notation $[-3, 0]$ and $[1, 3]$.

Exercises:

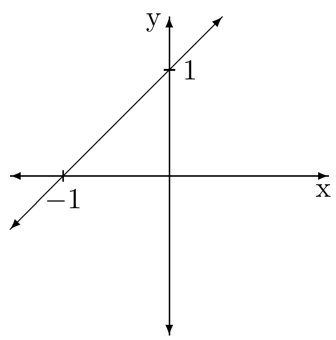
1. Find the range and domain of the following graphs



Section 2 Intercepts and Reading Graphs

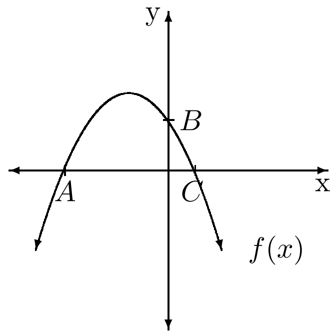
The value of a function when the input is zero (i.e. the y value when $x = 0$) is called the y-intercept. This is where the function crosses the y-axis. The input which gives the function a value of zero (i.e. the x values that give $y = 0$) are called the x-intercepts. They are called the x-intercepts because this is where the plot of the function crosses the x-axis. To find the y-intercept, we simply substitute $x = 0$ into the function. The output is the y-intercept. To find the x-intercept, we let $y = 0$ and then solve the equation for x. This may not always be a simple procedure. The intercepts give us the beginning of a picture of the function and will help us to represent the function graphically.

Example 1 :



The y-intercept is $+1$. The x-intercept is -1. i.e. when $x = 0$, $y = 1$ and when $y = 0$, $x = -1$.

Example 2 :

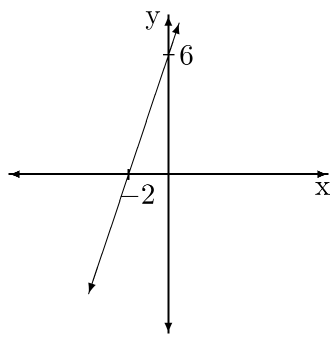


$f(x)$ has a zero value at A and C; B marks the y-intercept. There are many possible formulae for the graph, and one possibility is a parabola.

Example 3 : Let $y = 3x + 6$. Then at $x = 0$, $y = 3 \times 0 + 6 = 6$. So the y-intercept is 6. Now set $y = 0$ to find the x-intercept.

$$\begin{aligned} 0 &= 3x + 6 \\ x &= -2 \end{aligned}$$

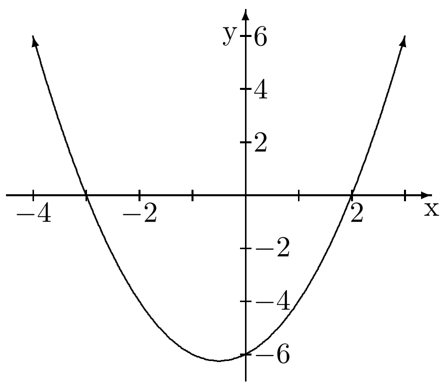
So the x-intercept is -2. Since $y = 3x+6$ is the equation for a straight line, we can now draw the graph. The intercepts of a straight line give us enough information to draw the graph (unless both intercepts are at the origin)



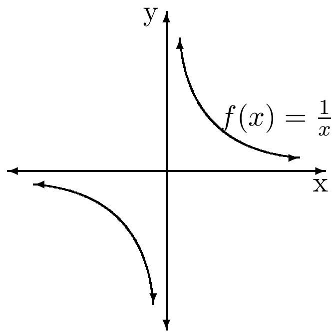
Example 4 : Let $y = x^2 + x + -6$. The y-intercept is -6 since $y = -6$ when $x = 0$. To find the x-intercepts, one must solve the equation

$$\begin{aligned} 0 &= x^2 + x - 6 \\ 0 &= (x + 3)(x - 2) \end{aligned}$$

This implies that we can have either $(x+3) = 0$ which gives $x = -3$ or $(x-2) = 0$ which gives $x = 2$. The x-intercepts are then -3 and 2. The equation we are currently dealing with is called a quadratic, and the intercepts don't give enough information to plot the graph. Since all quadratic functions are symmetrical, the turning point will always occur half-way between the x-intercepts. The equation $y = x^2 + x - 6$ is that of a parabola.

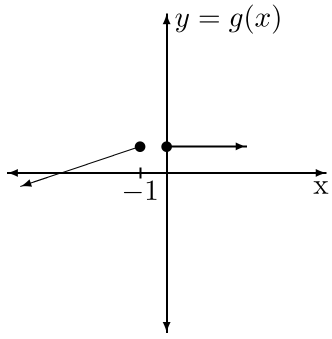


The worksheet on differentiation gives us another method for finding the coordinates of A. Point A is called a turning point because the graph changes direction at point A. Another important output of functions that can be seen from graphs are the values of y and x where the function isn't defined. These will appear as breaks in the graph. For example, we noted in the last worksheet that the function $y = \frac{1}{x}$ is not defined at $x = 0$. Also, there is no input that will give an output of $y = 0$, i.e. there are no x or y-intercepts. The graph of $y = \frac{1}{x}$ looks like this:



There is a break in the graph of this hyperbola at $x = 0$ and at $y = 0$.

Example 5 :



The function $g(x)$ is not defined for $-1 < x < 0$.

Exercises:

1. Graph the following equations of straight lines by first finding x and y intercepts.
   (a) $y = x + 2$
   (b) $y = x - 3$
   (c) $y = 2x + 4$
   (d) $x + y - 6 = 0$
   (e) $y = 4 - x$

Section 3 Odd and Even Functions

Some functions can be classed as odd or even functions. Many functions, however, are not odd or even. If we know that a certain function is odd or even, it will help us draw the graph.

An even function is one in which

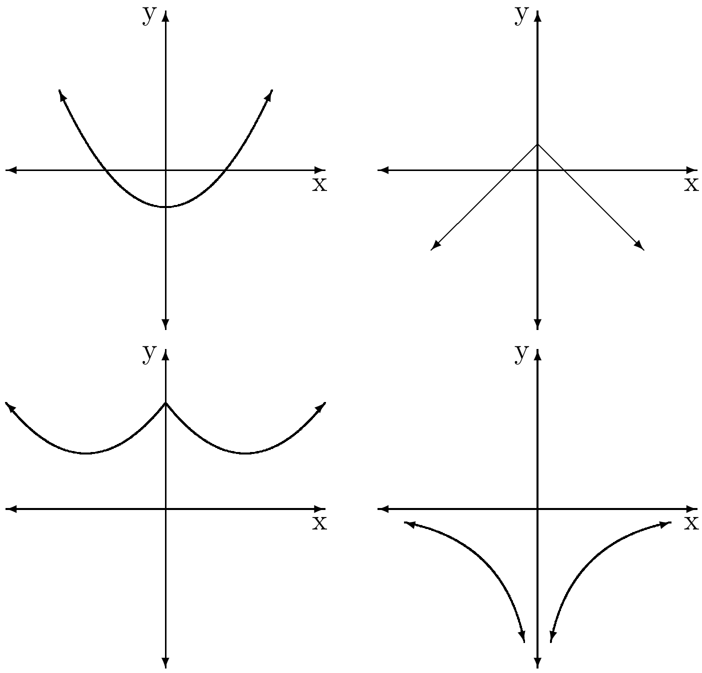
$$\boxed{f(a) = f(-a)}$$

for all a. That is,, whatever number we choose as input, the output of the function will be the same if we change the sign of the input. For an even function, if an input of $-c$ gives an output of b, then the input c also gives an output of b. In function notation, this says that if $f(-c) = b$, then $f(c) = b$ also.

Example 1 : The function $f(x) = x^2$ is even.

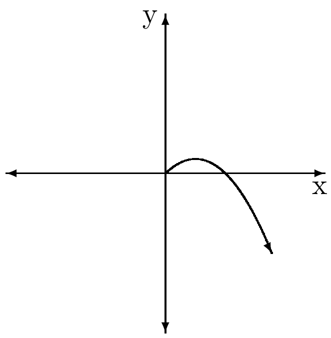
$$\begin{array}{ccc} f(1) = 1 & f(2) = 4 & f(a) = a^2 \\ f(-1) = 1 & f(-2) = 4 & f(-a) = a^2 \end{array}$$

An even function has a distinctive shape when graphed - the graph for the negative x's (the left-hand side of the y-axis) is a reflection of what is on the right-hand side of the y-axis.

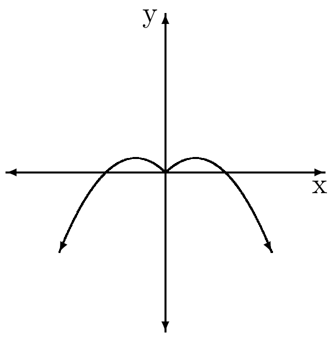


Each of the four graphs represents an even function.

Example 2 : Say we are given part of a graph of $g(x)$.



We now extend the graph of $g(x)$ to make an even function:



We now define what an odd function is. A function $f(x)$ is odd if

$$\boxed{f(-x) = -f(x)}$$

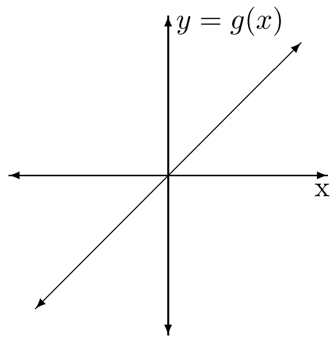
That is, if an input of a gives an output of b, then an input of $-a$ will give an output of $-b$.

Example 3 : The function $g(x) = x$ is an odd function. Note that

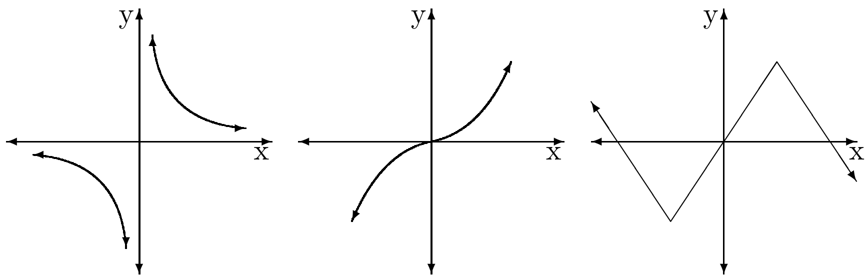
$$\begin{array}{cc} g(1) = 1 & g(2) = 2 \\ g(-1) = -1 & g(-2) = -2 \end{array}$$

The graph of an odd function has distinctive features; it has the property that you get what is on the right-hand side of the y-axis by rotating what is on the left-hand side of the y-axis through 180°. And vice versa.

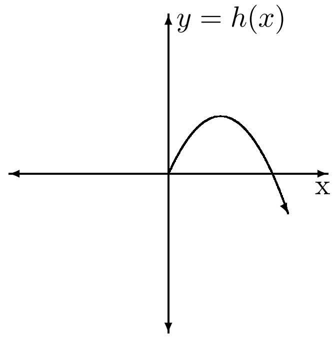
Example 4 :



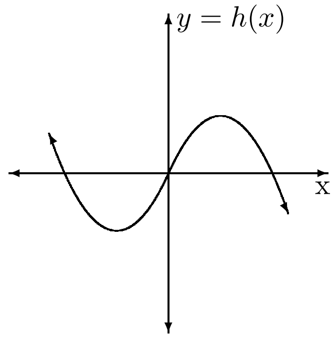
Here are more examples of odd functions.



Given the function $h(x)$:



We can extend $h(x)$ to be an odd function:



If we are given a function as an expression, we can test it to see if it is odd or even by substituting in a and $-a$ as inputs and finding out what the outputs are.

Example 5 : Is the function $y(x) = x^2 + 7$ even, odd, or neither?

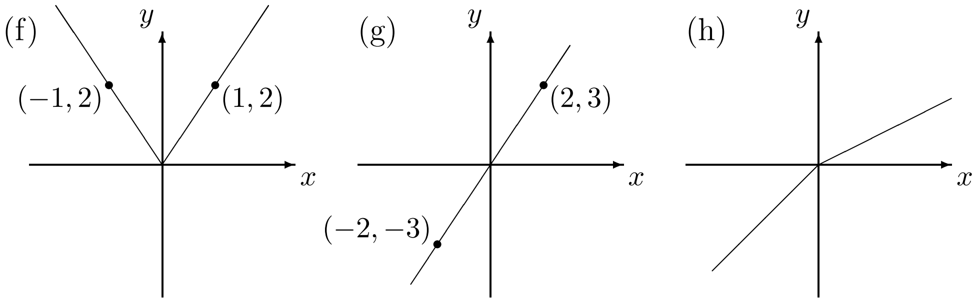
$$\begin{aligned} y(a) &= a^2 + 7 \\ y(-a) &= (-a)^2 + 7 \\ &= a^2 + 7 \end{aligned}$$

We can see that $y(-a) = y(a)$, so $y(x)$ is even.

Exercises:

1. Are the following functions even, odd, or neither?
   (a) $f(x) = 2x$
   (b) $f(x) = x^2 - 2$
   (c) $f(x) = x + 1$
   (d) $f(x) = \frac{3}{x}$

(e) $f(x) = x^2 + x + 3$



Section 4 The equation of a Circle

There are some equations that you should be able to recognize at first glance, and know roughly what they look like: the equation of a straight line is an example. Another equation that you should be able to recognize is the equation of a circle. It is

$$x^2 + y^2 = r^2$$

for some number r. When graphed, the set of points satisfying $x^2 + y^2 = r^2$ will be a circle of radius r centered at the origin. This means that the x and y-intercepts are $\pm r$.

The equation of a circle is not actually a function of x since each value of x has two possible values of y in the domain $-r < x < r$. But it is an equation that you will be expected to know how to graph.

A variation of the equation of a circle already given to you is

$$(x - a)^2 + (y - b)^2 = r^2$$

This is still an equation of a circle, but it is more general: it has a radius r as before, but is centred on (a, b) rather than at the origin.

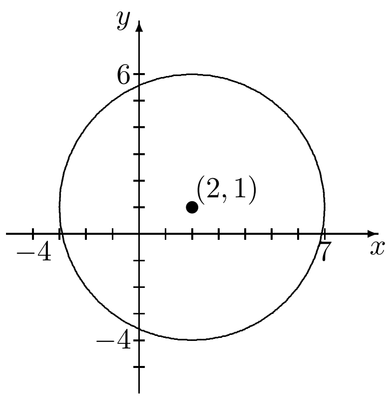
The circle

$$(x - 2)^2 + (y - 1)^2 = 25$$

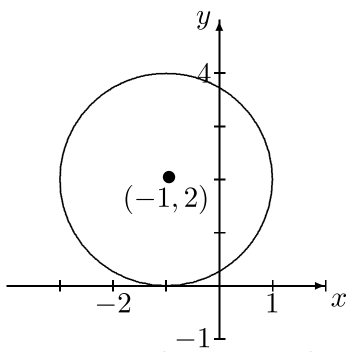
has centre $(2, 1)$ and radius 5. The circle

$$(x + 1)^2 + (y - 2)^2 = 4$$

has centre $(-1, 2)$ and radius 2.



$(x-2)^2+(y-1)^2=25$



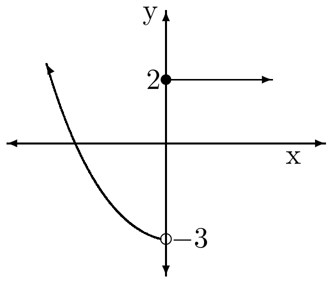
$(x+1)^2+(y-2)^2=4$

Exercises:

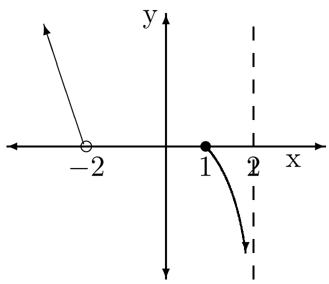
1. Write down the equation of a circle which has
   (a) radius 3 and centre $(6,-2)$
   (b) radius 5 and centre $(4,3)$
   (c) radius $1\frac{1}{2}$ and centre $(-1,2)$
   (d) radius 2 and centre $(\frac{1}{2},1\frac{1}{2})$
   (e) radius 4 and centre $(-1,-3)$

Exercises for Worksheet 3.2

1. (a) What is the range of the function whose graph is below? Give your answer in interval notation.



(b) What is the domain of the function whose graph is below (in interval notation)?



(c) What are the x and y intercepts of $y = 2x - 3$?

(d) The function $y = x^2 - 3x - 4$ crosses the x-axis twice and the y-axis once. Find all three intercepts.

(e) Let $f(x) = x^3 - x$.

i. Show that $f(x)$ is odd.

ii. Sketch $f(x)$.

2. State whether the following equations represent a line, a parabola, a cubic, a circle, or a hyperbola.

(a) $x^2 + y^2 = 16$

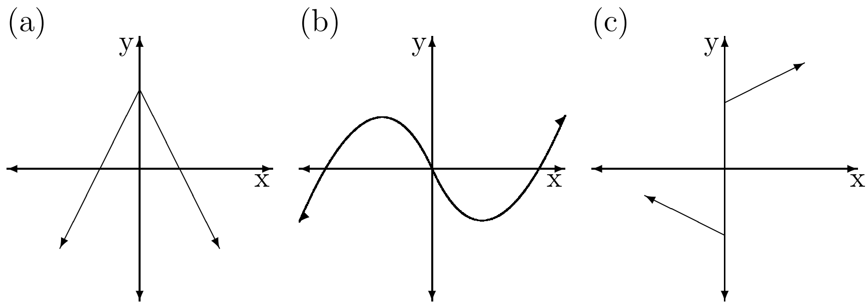
(b) $y = x^2 + 5x + 6$

(c) $y = \frac{2}{x}$

(d) $(x - 1)^2 + (y + 2)^2 = 49$

(e) $y = 2x + 1$

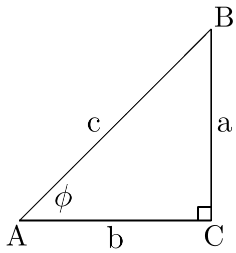
3. State whether the following functions are even, odd or neither:



Worksheet 3.3 Trigonometry

Section 1 Review of Trig Ratios

Worksheet 2.8 introduces the trig ratios of sine, cosine, and tangent. To review the ratios, consider a triangle ABC with angle ϕ as marked.



The hypotenuse (hyp) of the triangle is c; the adjacent (adj) side is b; the opposite (opp) side is a. The side of length a is opposite the angle, and the side of length b is the side adjacent to the angle which is *not* the hypotenuse. Then we have

$$\begin{aligned}\sin\phi &= \frac{\text{opp}}{\text{hyp}} = \frac{a}{c} \\ \cos\phi &= \frac{\text{adj}}{\text{hyp}} = \frac{b}{c} \\ \tan\phi &= \frac{\text{opp}}{\text{adj}} = \frac{a}{b}\end{aligned}$$

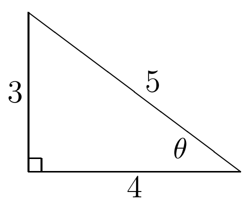
Note also that

$$\frac{\sin\phi}{\cos\phi} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} = \tan\phi$$

Exercises:

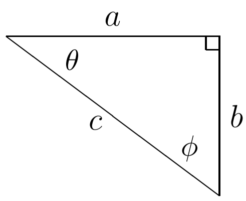
1. For the following triangle, find the ratios:

   (a) $\sin\theta$
   (b) $\tan\theta$
   (c) $\cos\theta$



2. For the following triangle, find the ratios:

(a)	$\tan\theta$	(d)	$\sin\phi$
(b)	$\cos\phi$	(e)	$\tan\phi$
(c)	$\sin\theta$	(f)	$\cos\theta$

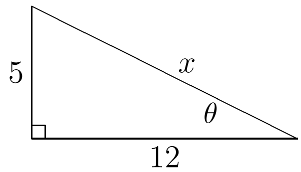


3. (a) Use Pythagoras' theorem to find x

   (b) Find (i) $\sin\theta$

   (ii) $\tan\theta$

   (iii) $\cos\theta$



Section 2 Degrees and Radians

Recall from Worksheet 2.9 that

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

In university maths it is much more common to give angles in radians rather than degrees. If the units are left off an angle, then the angle is in radians. Degrees can be converted to radians using the above formula, but it will be very convenient for you to know some standard conversions. In particular:

$$\begin{array}{cc} 90^\circ = \frac{\pi}{2} & 30^\circ = \frac{\pi}{6} \\ 45^\circ = \frac{\pi}{4} & 180^\circ = \pi \\ 60^\circ = \frac{\pi}{3} & 360^\circ = 2\pi \end{array}$$

Example 1 : An equilateral △ has three equal angles of $\frac{\pi}{3}$.

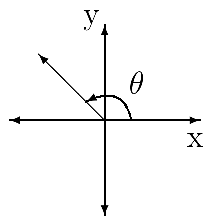
Example 2 : Convert 50° to radians.

$$50^\circ = 50\frac{\pi}{180} = \frac{5\pi}{18} \text{ radians}$$

Example 3 : How many degrees is $\frac{\pi}{9}$ radians? We know $180^\circ = \pi$, so $\frac{\pi}{9} = \frac{180^\circ}{9}$ therefore

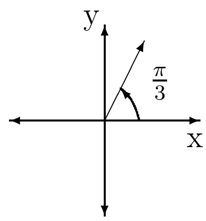
$$\frac{\pi}{9} = 20^\circ$$

If we think of an angle θ as the amount of rotation of a straight line about the angle, then we can define a positive rotation as one which is anti-clockwise and a negative rotation as one which is in a clockwise direction. We can see the ordinary $x-y$ plane with the vertex of the angle at the origin and the base of the angle beginning at the positive x-axis. i.e. the positive x-axis represents the position of the line if the angle of rotation is 0. So for the angle θ we get the following diagram:

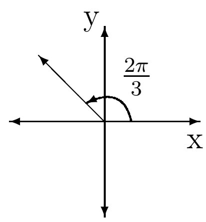


We can now represent angles graphically and we can deal with angles of any size. Recall that a full revolution is 2π, or $360°$. So rotating a line through 2π will bring it back to its starting position.

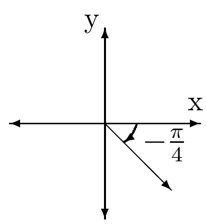
Example 4 : Represent $\frac{\pi}{3}$ radians graphically. Since π is the angle half way round the plane, $\frac{\pi}{3}$ is the angle one third of the way around the upper half of the plane.



Example 5 : Represent $\frac{2\pi}{3}$ radians graphically.



Example 6 : Represent $-\frac{\pi}{4}$ radians graphically.



Exercises:

1. Write the following degrees in radian measure

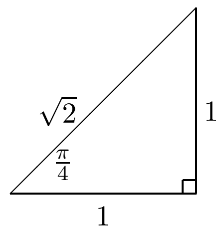
   (a) $80°$ (b) $120°$ (c) $90°$ (d) $270°$ (e) $135°$

2. Convert the following radian measures to degrees

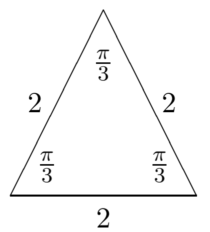
   (a) $\frac{\pi}{8}$ (b) $\frac{2\pi}{9}$ (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{6}$ (e) $\frac{7\pi}{6}$

Section 3 Standard Triangles

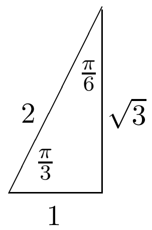
There are two triangles which are known as standard triangles. These triangles and the information contained in them should be memorized, as you will be expected to know certain information without using a calculator. The first triangle is a right-angled isosceles triangle. Recall that an isosceles triangle has two angles the same and two sides the same length:



The associated trig ratios are fairly simple to work out, and are left as exercises. The second standard triangle is half an equilateral triangle of length 2.



When we have taken half the equilateral triangle, we end up with the following:



Pythagoras' theorem gives us the length of the vertical side as $\sqrt{3}$, and the angle ϕ is half the top angle so $\phi = \frac{\pi}{6}$. The trig ratios given by this triangle are:

$$
\begin{aligned}
\sin\tfrac{\pi}{3} &= \tfrac{\sqrt{3}}{2} & \sin\tfrac{\pi}{6} &= \tfrac{1}{2}\\
\cos\tfrac{\pi}{3} &= \tfrac{1}{2} & \cos\tfrac{\pi}{6} &= \tfrac{\sqrt{3}}{2}\\
\tan\tfrac{\pi}{3} &= \sqrt{3} & \tan\tfrac{\pi}{6} &= \tfrac{1}{\sqrt{3}}
\end{aligned}
$$

Once the triangles are memorized the trig ratios can be found by just drawing either of the two triangles. It is important that you memorize the trig ratios for the angles $\frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{\pi}{4}$.

Exercises:

1. Find the exact ratios for the following

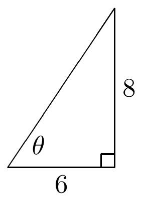
   (a) $\tan\frac{\pi}{3}$ (b) $\cos\frac{\pi}{6}$ (c) $\sin\frac{\pi}{4}$ (d) $\tan\frac{\pi}{4}$ (e) $\tan\frac{\pi}{6}$

2. Use exact ratios to find θ in each of the following equations, where $0 \leq \theta \leq \frac{\pi}{2}$.

   (a) $\sin\theta = \frac{1}{2}$
   (b) $\cos\theta = \frac{1}{2}$
   (c) $\tan\theta = 1$
   (d) $\sin\theta = \frac{\sqrt{3}}{2}$
   (e) $\cos\theta = \frac{1}{\sqrt{2}}$

Section 4 Using trigonometric ratios

We can use the trigonometric ratios described in the previous sections to find an unknown angle or side in a right-angled triangle. Consider the following triangle:

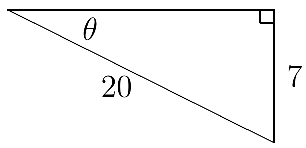


Let us say that we wish to find θ in this triangle. The side that is 6 units long is adjacent to θ; the side that is 8 units long is opposite to θ, so we have

$$
\begin{aligned}
\tan\theta &= \frac{\text{opposite}}{\text{adjacent}}\\
&= \frac{8}{6}\\
&= \frac{4}{3}
\end{aligned}
$$

Consequently, θ is an angle whose tangent is $\frac{4}{3}$. That is $\theta = \tan^{-1}\frac{4}{3}$. By using the $\tan^{-1}$ button on a calculator, we find that $\theta = 0.927$, to three decimal places. Note that this answer is in radians.

<u>Example 1</u> : What is θ?

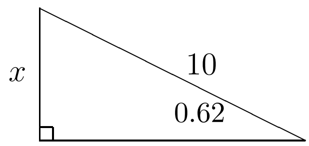


With respect to θ, the opposite side is 7 units long, and the hypotenuse is 20 units long. Therefore,

$$\begin{aligned} \sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\ &= \frac{7}{20} \\ \theta &\doteq 0.358 \end{aligned}$$

The last step was carried out using the $\sin^{-1}$ button on a calculator, and the answer is approximate and in radians.

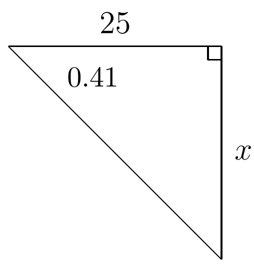
<u>Example 2</u> : What is x?



The trigonometric ratios may also be used to find the length of a side in a right angled triangle.

$$\begin{aligned} \sin 0.62 &= \frac{\text{OPP}}{\text{HYP}} \\ &= \frac{x}{10} \\ x &= 10\sin 0.62 \\ x &= 5.81 \end{aligned}$$

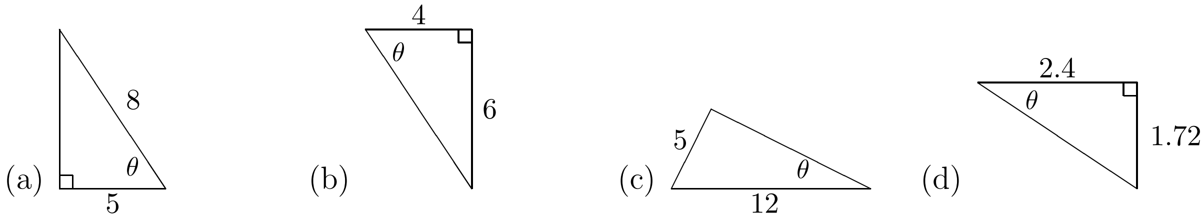
<u>Example 3</u> : What is x?



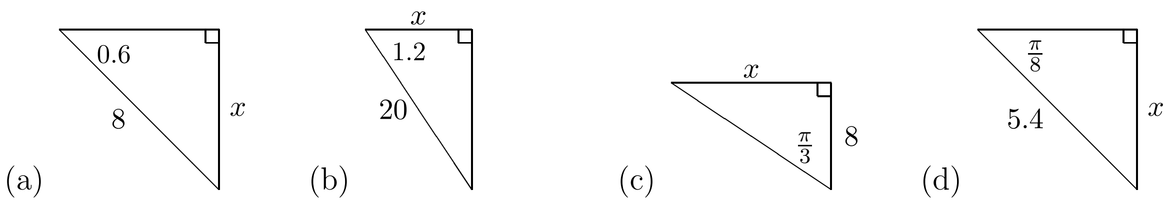
$$
\begin{aligned}
\tan 0.41 &= \frac{\text{OPP}}{\text{ADJ}} \\
&= \frac{x}{25} \\
x &= 25 \tan 0.41 \\
x &= 10.87
\end{aligned}
$$

Exercises:

1. Find θ in each of the following



2. Find x, to 2 decimal places, in each of the following triangles.



Section 5 Inverse Trig Functions

Sometimes you will come across the notation $\sin^{-1} a$ or $\cos^{-1} a$. Now, $\sin^{-1} a$ does not mean $\frac{1}{\sin a}$. It is called the arcsine of a, and means this: the sin of what angle will give an answer a? So

$$
\begin{aligned}
\sin^{-1} a &= \theta \quad \text{means} \\
\sin \theta &= a
\end{aligned}
$$

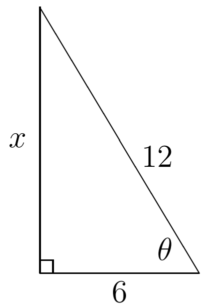
The same rule applies to $\cos^{-1} a$ and $\tan^{-1} a$. If you wish to write $\frac{1}{\sin a}$ then you would do so as $(\sin a)^{-1}$ so that there is no confusion.

Exercises for Worksheet 3.3

1. (a) Covert
   i. $\frac{5\pi}{6}$ to degrees
   ii. $\frac{12\pi}{9}$ to degrees
   iii. 80° to radians; write the answer as a number times π.
   iv. 42° to radians; write the answer to 3 decimal places

   (b) Find the exact values of
   i. $\sin\frac{\pi}{4}$
   ii. $\cos\frac{\pi}{6}$
   iii. $\tan\frac{\pi}{3}$

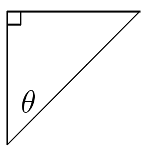
2.



   (a) Find the value of x
   (b) Evaluate θ

3. Joan walks 5km north, then 3.6km east.

   (a) Put these distances onto the appropriate sides of the triangle below:



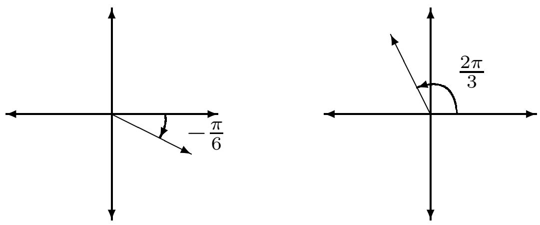
   (b) Find the angle θ, the bearing that Joan has effectively walked along.

Worksheet 3.4 Further Trigonometry

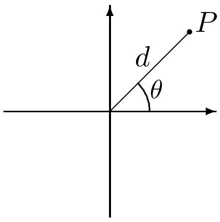
Section 1 Trig ratios for angles of any magnitude

Recall from the last worksheet how we described a way of drawing angles of any magnitude on the cartesian plane. If we use the positive x-axis to represent our starting point, then rotate this axis in an anticlockwise direction through α radians, we have an angle of α radians (with α positive). A negative angle can be drawn by rotation in a clockwise direction.

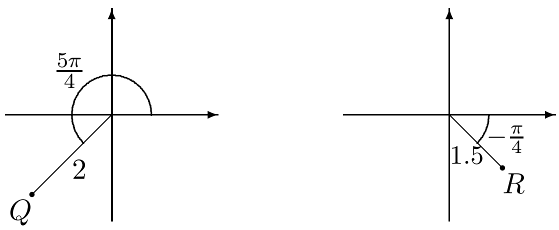
Example 1 : We draw the angles $-\frac{\pi}{6}$ and $\frac{2\pi}{3}$.



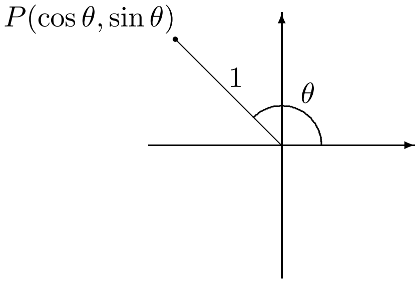
Instead of representing a point P using x and y coordinates, we could represent it as an angle of rotation and a distance away from the origin.



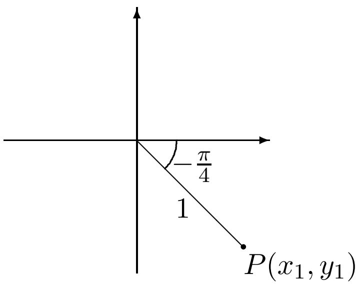
As examples consider two points: Q, which is 2 units away from the origin and rotated through an angle of $\frac{5\pi}{4}$; R, which is 1.5 units away from the origin and rotated through an angle of $-\frac{\pi}{4}$.



The question now is how to connect this method of specifying a point with the usual way of using the x and y coordinates. In a previous section we defined $\sin\theta$ and $\cos\theta$ using right angled triangles. A much more useful definition is the following. Let a point P be exactly one unit away from the origin, and rotated by an angle θ. Then the x coordinate of P is defined to be $\cos\theta$, and the y coordinate is defined to be $\sin\theta$. The illustration of the definition is:



Example 2 : Draw a picture to determine whether $\sin(-\frac{\pi}{4})$ and $\cos(-\frac{\pi}{4})$ are positive or negative.

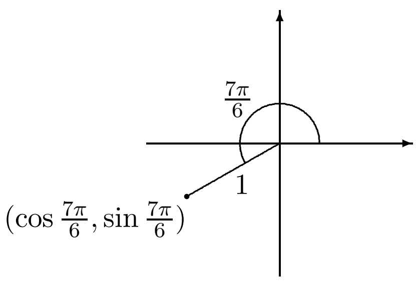


From the definitions of sine and cosine we have

$$
\begin{aligned}
x_1 &= \cos(-\frac{\pi}{4}) \\
y_1 &= \sin(-\frac{\pi}{4})
\end{aligned}
$$

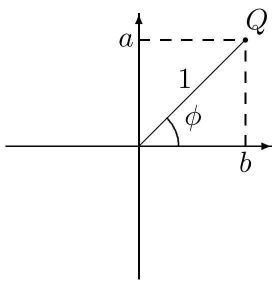
It is apparent from the picture that $\cos(-\frac{\pi}{4}) > 0$ and $\sin(-\frac{\pi}{4}) < 0$.

Example 3 : Draw a picture to locate the point $(\sin(\frac{7\pi}{6}), \cos(\frac{7\pi}{6}))$.



Notice what happens when we apply this definition to an angle that is between 0 and $\frac{\pi}{2}$. Let Q be the point that is exactly one unit away from the origin, and rotated by an angle ϕ, where

$0 < \phi < \frac{\pi}{2}$. Say the x coordinate of Q is b and the y coordinate is a. The relevant picture is:

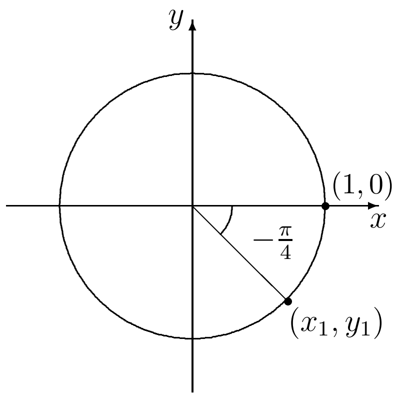


If you think about the right angled triangle formed be the points $(0,0)$, $(b,0)$, and (a,b) and apply the right angled definitions of sine and cosine then you get the formulae

$$\begin{aligned} \sin\phi &= \frac{a}{1} = a \\ \cos\phi &= \frac{b}{1} = b \end{aligned}$$

which is to say the x and y coordinates of the point Q are $\cos\phi$ and $\sin\phi$ respectively. This is exactly the definition that we have just proposed! The point is that the definitions of sine and cosine that we have seen before, in terms of right angled triangles, match the new definition that we have just given in the case that the angles are between 0 and $\frac{\pi}{2}$ ($0°$ and $90°$).

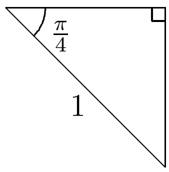
The advantage with the new definition is that it allows us to find the sine and cosine of angles of any magnitude, as well as for negative angles. We do this by drawing the unit circle (which is a circle of radius 1 centred on the origin). Any point on the circle is then exactly one unit away from the origin. Now, drawing in our angle from example 2, $-\frac{\pi}{4}$, we get



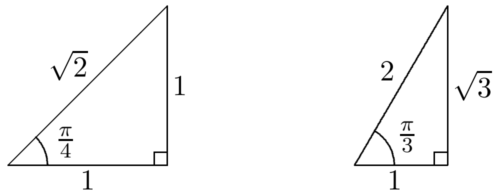
Now,

$$\begin{aligned} x_1 &= \cos(-\frac{\pi}{4}) \\ y_1 &= \sin(-\frac{\pi}{4}) \end{aligned}$$

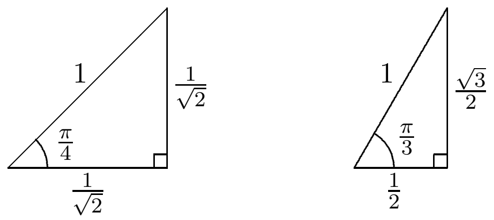
Notice that the triangle defined by the points $(0,0)$, $(x_1,0)$ and (x_1,y_1) is a right angled triangle; the hypotenuse is of length 1 because the the radius of the unit circle is of length 1. It is drawn here:



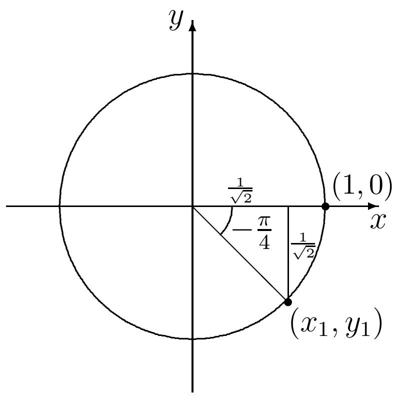
What remains is to find the length of the two sides of the triangle, which we will do by recalling our standard triangles from an earlier worksheet and using the properties of similar triangles. The two standard triangles we looked at were



We can make the triangle on the left the same as the one that we took out of the unit circle by dividing all the lengths by a factor of $\sqrt{2}$. Similarly, we can make the standard triangle on the right have a hypotenuse of length 1 by dividing each side by 2.



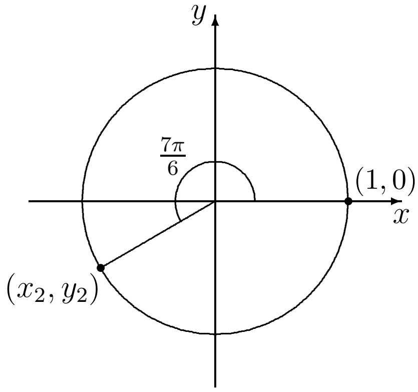
Putting the lengths back onto the unit circle picture gives us



Then we have

$$
\begin{aligned}
x_1 &= \cos(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \\
y_1 &= \sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}
\end{aligned}
$$

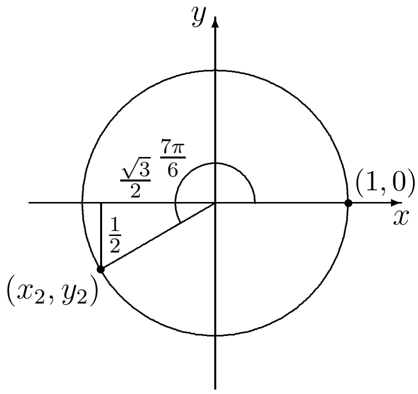
Example 4 : Calculate the sine and cosine of $\frac{7\pi}{6}$.



Recalling the definitions of sine and cosine, we have

$$\begin{aligned} x_2 &= \cos(\frac{7\pi}{6}) \\ y_2 &= \sin(\frac{7\pi}{6}) \end{aligned}$$

By extracting the right angled triangle which connects the points $(0, 0)$, $(x_2, 0)$, and (x_2, y_2) and comparing it to the scaled standard triangles, we can put the following distances onto the unit circle diagram.



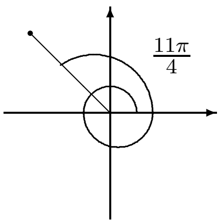
As a result, we get

$$\begin{aligned} x_2 &= \cos(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2} \\ y_2 &= \sin(\frac{7\pi}{6}) = -\frac{1}{2} \end{aligned}$$

What happens if an angle is bigger than 2π or less than -2π? Since a full revolution of a circle is 2π radians, the position on the circle is unchanged if we go an angle of θ or an angle of

$\theta + 2\pi$. If the position on the circle is unchanged by adding an angle of 2π, and the sines and cosines are defined in terms of the coordinates of appropriate points on the circle, then the sines and cosines of angles are unchanged by adding or subtracting multiples of 2π radians.

As an example, graph the angle $\frac{11\pi}{4}$ on the cartesian plane.



Notice that we would end up with an angle pointing in the same direction if we had performed a rotation of $\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}$.

Exercises:

1. Find the exact ratio for each of the following

   (a) $\sin\frac{\pi}{4}$ (b) $\tan(-\frac{\pi}{4})$ (c) $\cos\frac{\pi}{6}$ (d) $\cos\frac{7\pi}{6}$ (e) $\sin(-\frac{\pi}{3})$ (f) $\tan\frac{3\pi}{4}$ (g) $\cos\frac{3\pi}{4}$ (h) $\sin(-\frac{2\pi}{3})$ (i) $\tan\frac{5\pi}{6}$ (j) $\cos(-\frac{\pi}{3})$

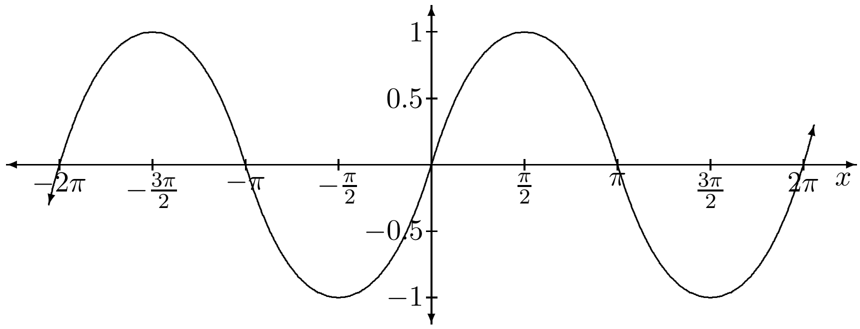
2. Use a calculator to find the following to 2 decimal places.

   (a) $\cos 1.6$ (b) $\sin\frac{\pi}{8}$ (c) $\tan\frac{7\pi}{6}$ (d) $\cos(-\frac{\pi}{7})$ (e) $\sin(-0.6)$ (f) $\sin\frac{\pi}{9}$ (g) $\cos\frac{8\pi}{5}$ (h) $\tan(-\frac{\pi}{9})$

Section 2 Graphs of Trig Functions

The trig functions can be graphed on a Cartesian plane as functions of x. The unit of measurement for x is radians. It is helpful to be able to recognize the graphs of the main trig functions.

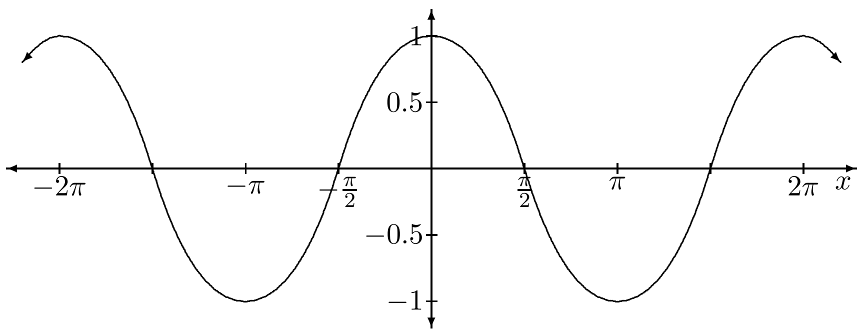
The function $y = \sin x$ is odd, with an x-intercept every integer multiple of π. It looks like this:



The function $y = \sin x$ is also periodic, with period 2π. This means that $\sin(x) = \sin(x + 2\pi)$ for all values of x. If a function $f(x)$ is periodic with period b, then $f(x) = f(x + b)$ for all x.

We can see from the graph of $y = \sin x$ that the range of the function is $[-1, 1]$ i.e. $-1 \leq \sin x \leq 1$ for all x.

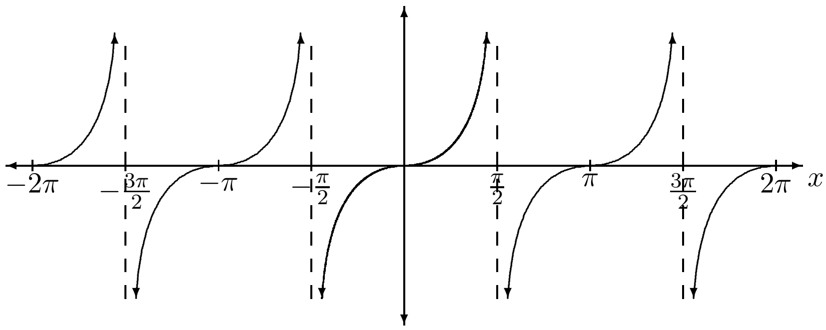
The function $y = \cos x$ is even. It is also periodic with period 2π. The y-intercept is 1 and the x-intercepts are at $\frac{\pi}{2} + k\pi$ for integer k. The graph of $y = \cos x$ looks like this:



Notice that the range is also $[-1, 1]$, so $-1 \leq \cos x \leq 1$ for all x. The graphs of $\sin x$ and $\cos x$ will help you to remember the values of $\sin x$ and $\cos x$ for $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π if you are a visual person. Some people find it easier to remember the pictures, other people the numbers. From the graph we get

$$\begin{aligned} \sin(-\pi) &= 0 & \cos(-\pi) &= -1 \\ \sin(-\tfrac{\pi}{2}) &= -1 & \cos(-\tfrac{\pi}{2}) &= 0 \\ \sin(\tfrac{\pi}{2}) &= 1 & \cos(\tfrac{\pi}{2}) &= 0 \\ \sin(\tfrac{3\pi}{2}) &= -1 & \cos(\tfrac{3\pi}{2}) &= 0 \\ \sin(2\pi) &= 0 & \cos(2\pi) &= 1 \end{aligned}$$

The graph of $y = \tan x$ looks completely different from either $\cos x$ or $\sin x$. It is a periodic function with period π and it looks like this:



Notice that the x-intercepts are integer multiples of π, and that the y-intercept is 0. Notice also that $y = \tan x$ is not defined at $\frac{\pi}{2} + k\pi$ for any integer k. Recall that

$$\tan x = \frac{\sin x}{\cos x}$$

so $\tan x$ is undefined when $\cos x = 0$, which is at $\frac{\pi}{2} + k\pi$ for any integer k.

Exercises:

1. (a) Given $y = 2\sin x$, complete the table of values

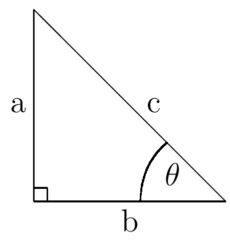
x	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π
y									

   (b) Using the table draw the graph of $y = 2\sin x$ for $-\pi \le x \le \pi$.

Section 3 Pythagorean Identities

There are some equalities known as trigonometric identities which are very useful in solving some kinds of problems. The first one that we look at is derived from Pythagoras' theorem. Recall:

$$\begin{aligned}\sin\theta &= a/c\\ \cos\theta &= b/c\\ a^2+b^2 &= c^2\end{aligned}$$



From the above relations, we then have:

$$\begin{aligned}\sin^2\theta+\cos^2\theta &= (\frac{a}{c})^2+(\frac{b}{c})^2\\ &= \frac{a^2}{c^2}+\frac{b^2}{c^2}\\ &= \frac{a^2+b^2}{c^2}\\ &= \frac{c^2}{c^2}\\ &= 1\end{aligned}$$

Then we have that for any angle θ:

$$\boxed{\sin^2\theta+\cos^2\theta=1}$$

The next two identities are also important, but will not be derived. For any angles A and B:

$$\boxed{\begin{aligned}\sin(A+B) &= \sin A\cos B+\sin B\cos A\\ \cos(A+B) &= \cos A\cos B-\sin A\sin B\end{aligned}}$$

These identities can be used to find the cos and sin of any angles bigger than $\frac{\pi}{2}$. We can derive more trig identities from the ones that we already have.

Example 1 :

$$\begin{aligned}\sin 2x &= \sin(x+x)\\ &= \sin x\cos x+\sin x\cos x\\ &= 2\sin x\cos x\end{aligned}$$

Example 2 :

$$\begin{aligned}\cos 2x &= \cos(x+x)\\ &= \cos x\cos x - \sin x\sin x\\ &= \cos^2 x - \sin^2 x\end{aligned}$$

Recall that $y = \cos x$ is an even function, therefore

$$\cos(-x) = \cos(x)$$

Recall that $y = \sin x$ is an odd function, therefore

$$\sin(-x) = -\sin(x)$$

Example 3 :

$$\begin{aligned}\sin(A-B) &= \sin A\cos(-B) + \sin(-B)\cos A\\ &= \sin A\cos B - \sin B\cos A\end{aligned}$$

Example 4 :

$$\begin{aligned}\cos(A-B) &= \cos A\cos(-B) - \sin(-B)\sin A\\ &= \cos A\cos B + \sin B\sin A\end{aligned}$$

These identities can be used in many ways. One use for them is an alternative way of finding the trig ratios of angles between 0 and 2π.

Example 5 : Calculate the sine and cosine of $\frac{7\pi}{6}$.

$$\begin{aligned}\sin\frac{7\pi}{6} &= \sin(\pi+\frac{\pi}{6})\\ &= \sin\pi\cos\frac{\pi}{6} + \cos\pi\sin\frac{\pi}{6}\\ &= 0\times\frac{\sqrt{3}}{2} + (-1)\times\frac{1}{2}\\ &= -\frac{1}{2}\\ \cos\frac{7\pi}{6} &= \cos(\pi+\frac{\pi}{6})\\ &= \cos\pi\cos\frac{\pi}{6} - \sin\pi\sin\frac{\pi}{6}\\ &= (-1)\times\frac{\sqrt{3}}{2} + (0)\times\frac{1}{2}\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Writing the angle $\frac{7\pi}{6}$ as $\pi + \frac{\pi}{6}$ wasn't the only option – we could have used $\frac{7\pi}{6} = \frac{3\pi}{2} - \frac{\pi}{3}$. (Notice that the answers that we have here agree with the values calculated using the unit circle earlier in the worksheet.)

Exercises:

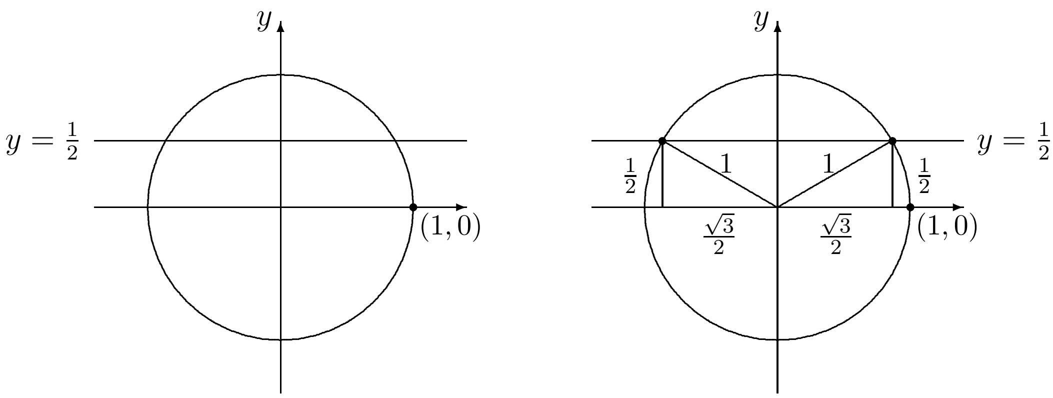
1. Use exact ratios to show that
$$\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{6} = 1$$

2. Use exact values to show that equation
$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$
is satisfied when $A = 0$ and $B = \frac{\pi}{3}$.

Section 4 Solving trigonometric equations

In the previous section θ was given and we evaluated the trigonometric ratios for the angle. Now we investigate the situataion where we must find the value, or values, of θ when we are given a trigonometric ratio.

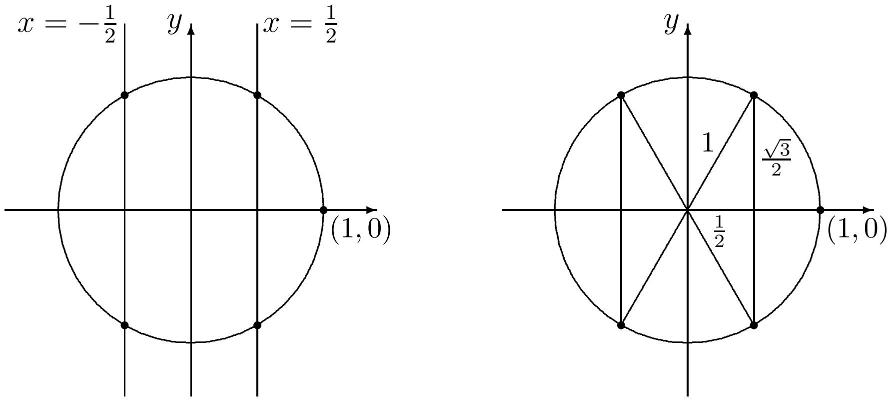
Example 1 : Solve $\sin\theta = \frac{1}{2}$ for $0 \le \theta \le 2\pi$. Recall that $\sin\theta$ is the y coordinate of a point on the unit circle. The first step then will be to draw a unit circle and draw the line $y = \frac{1}{2}$; the next, and last, step is to determine the angles of the points on the unit circle where the line $y = \frac{1}{2}$ cuts. We draw two pictures, one with the basic information we have just outlined, and one with a few distances that have been worked out.



The first thing to note is that there are two solutions. The lengths shown have been figured out using the fact that the vertical distances are $\frac{1}{2}$, the fact that the radius of the circle is 1, and by recognizing that the triangles hidden in the picture are scaled versions of the standard triangles (which are shown in section 1). Given that we know the angles in the standard triangles, we can read off the angles to the two solutions as $\frac{\pi}{6}$ and $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Example 2 : Solve $\cos^2\theta = \frac{1}{4}$.

This must have solutions given by $\cos\theta = \frac{1}{2}$ and $\cos\theta = -\frac{1}{2}$. The definition of $\cos\theta$ is that it is the x coordinate of the point defined by some angle around the unit circle. So the solution will be obtained by drawing the lines $x = \frac{1}{2}$ and $x = -\frac{1}{2}$, locating the points of intersection with the unit circle, then finding the appropriate angles. Again we draw two pictures: one with minimal information, so we can see roughly where the solutions are as well as how many solutions there are; the other picture has details of distances and so on.



From the first picture we can see that there are four solutions, as well as the fact that there is one solution in each quadrant. Of the four triangles in the second picture, only the distances for one of them have been shown, as all the other triangles are similar. The distances have again been found by scaling a standard triangle from section 1. As the angles of the standard triangle are known, so are the angles of the four points shown. They are $\frac{\pi}{3}$, $\pi - \frac{\pi}{3}$, $\pi + \frac{\pi}{3}$, and $2\pi - \frac{\pi}{3}$. (Another way to write the solutions would be $\frac{\pi}{3}$, $\pi - \frac{\pi}{3}$, $-\frac{\pi}{3}$, and $-(\pi - \frac{\pi}{3})$.)

Exercises:

1. Solve the following equations for θ; restrict your answers to $0 \leq \theta \leq 2\pi$.

   (a) $\sin\theta = -\frac{\sqrt{3}}{2}$

   (b) $\tan\theta = -\sqrt{3}$

   (c) $\cos\theta = \frac{1}{\sqrt{2}}$

   (d) $\cos\theta = \frac{\sqrt{3}}{2}$

   (e) $\tan\theta = -1$

   (f) $\sin\theta = -\frac{1}{2}$

Exercises for Worksheet 3.4

1. Find the exact ratios of

   (a) $\sin \frac{3\pi}{4}$

   (b) $\tan \frac{\pi}{6}$

   (c) $\cos \frac{7\pi}{4}$

   (d) $\cos \frac{4\pi}{3}$

   (e) $\sin \frac{5\pi}{6}$

   (f) $\tan -\frac{\pi}{3}$

2. (a) Use the expansion $\sin(A+B) = \sin A \cos B + \sin B \cos A$ to find the exact value of $\sin \frac{7\pi}{12}$. Note that $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$.

   (b) Use the expansion $\cos(A-B) = \cos A \cos B + \sin A \sin B$ to find the exact value of $\cos \frac{\pi}{12}$. Note that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.

3. Solve for θ in the interval $0 \leq \theta \leq 2\pi$.

   (a) $\tan \theta = \sqrt{3}$

   (b) $\sin \theta = \frac{1}{\sqrt{2}}$

   (c) $\cos \theta = -\frac{1}{\sqrt{2}}$

Worksheet 3.5 Simultaneous Equations

Section 1 Number of Solutions to Simultaneous Equations

In maths we are sometimes confronted with two equations in two variables and we want to find out which values of the variables satisfy both of the equations. Sometimes there will be no values of the variables that allow both equations to hold, and other equations will have many possible values of the variables. The process of finding solutions is called solving simultaneous equations .

For example, we might be asked to find the x and y values that allow both of the following equations to be true:

$$\begin{aligned} 5x + 2y &= 3 \\ 4x + 2y &= 4 \end{aligned}$$

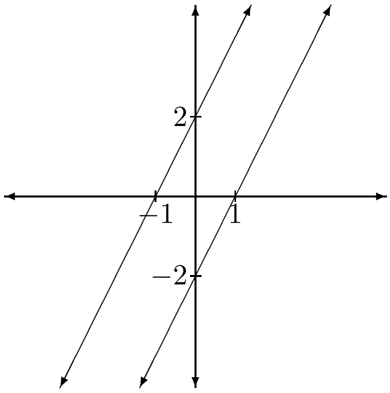
Recall from worksheet 2.10 that the general equation of a line can be written in the form $ax + by + c = 0$, for a, b, and c constants. Both of the equations above have the correct form for the equation of a line. This is always the case when solving linear simultaneous equations in two variables. This means that solving simultaneous equations is the same as finding the point of intersection of lines. If certain values of x and y satisfy both equations, the point (x, y) will lie on both the lines. If we think about a system of simultaneous equations as representing lines on the cartesian plane we can tell how many solutions there will be to the equations without actually solving them. When we draw two lines on the plane, there are three possibilities:

1. The lines cross just once.
2. The lines never cross.
3. The lines lie on top of each other.

The first case corresponds to a unique solution, i.e. there is only one value for each variable that will satisfy both equations. The second case occurs when the two lines are parallel, and aren't touching. Parallel lines have the same slope (gradient). For instance, the two lines

$$\begin{aligned} 1.\quad y &= 2x + 2 \quad \text{and} \\ 2.\quad y &= 2x - 2 \end{aligned}$$

are parallel, and don't touch. They have the same gradient, in this case 2, but the y-intercepts are different. Consequently, they two lines never touch each other. Equation 1 has a y-intercept of 2, and equation 2 has a y-intercept of -2. If we draw these lines, we get



The third case arises when the two equations represent the same line on the plane, and so touch everywhere. The two equations

1. $y = 3x + 6$ and
2. $2y = 6x + 12$

lie on top of one another when graphed because if we take equation 2 and divide both sides by 2, they we get equation 1 exactly. If two lines lie on top of one another there are an infinite number of (x, y) pairs that will satisfy both equations. Namely, every pair (x, y) that satisfies equation 1 will also satisfy equation 2.

To check on the number of solutions to a system of simultaneous equations we can rearrange both equations to the slope-intercept form and then compare gradients and intercepts. If the gradients are different, we will have a single (i.e. unique) solution. If the gradients are the same, but the y-intercepts are different, then we will have no solutions. If the gradients are the same, and the y-intercepts are the same, then there will be an infinite number of solutions.

Putting this algebraically, if we have:

$$\begin{aligned} y &= m_1 x + b_1 \\ y &= m_2 x + b_2 \end{aligned}$$

Then

- If $m_1 \neq m_2$ then there is one solution.
- If $m_1 = m_2$ and $b_1 \neq b_2$ there are no solutions.
- If $m_1 = m_2$ and $b_1 = b_2$ there is an infinite number of solutions

Example 1 : How many solutions do the following simultaneous equations have?

1. $3y + 6x = 9$
2. $2y + 10x = 4$

Rearranging, we get

$$\begin{aligned} y &= -2x + 3 \quad \text{for equation 1} \\ y &= -5x + 2 \quad \text{for equation 2} \end{aligned}$$

The gradients of the two lines are different, so there will be one solution.

Example 2 : How many solutions do the following simultaneous equations have?

1. $5y + 10x = 5$
2. $y + 2x = 2$

Rearranging, we get

$$\begin{aligned} y &= -2x + 1 \quad \text{for equation 1} \\ y &= -2x + 2 \quad \text{for equation 2} \end{aligned}$$

The gradients of the two lines are the same, but the intercepts are different. Then the lines are parallel, but don't touch. There are no solutions to the system.

Example 3 : How many solutions do the following simultaneous equations have?

1. $5 + 10x = 2y$
2. $4y - 20x = 10$

Rearranging, we get

$$\begin{aligned} y &= \frac{5}{2} + 5x \quad \text{for equation 1} \\ y &= \frac{5}{2} + 5x \quad \text{for equation 2} \end{aligned}$$

The gradients of the two lines are the same, and the intercepts are also the same. Then the lines are on top of each other, and there are infinitely many solutions.

Exercises:

1. How many solutions would each of the following pairs of equations have?

   (a) $y = 2x + 1$
   $y = 3x - 2$

   (b) $2y = 6x - 4$
   $y = 3x - 2$

   (c) $y + x - 2 = 0$
   $y - x + 1 = 0$

2. Check your answers by graphing the pairs of lines on a number plane.

Section 2 Solving simultaneous equations

The previous section discussed how many solutions there are to a system of 2 simultaneous equations in 2 unknowns (which we have been writing as x and y). We will learn how to find solutions to a system of simultaneous equations by example. Given two equations and two unknowns, our objective is to reduce this to one equation and one unknown, which we know we can solve.

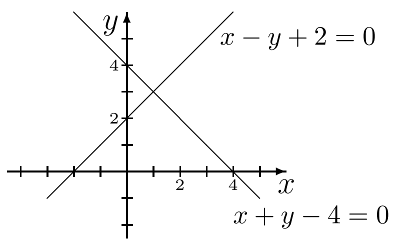
We can solve two equations simultaneously by graphing them and finding their point of intersection.

Let us solve for the following system graphically:

(i) $x + y - 4 = 0$

(ii) $x - y + 2 = 0$

Drawing the two lines on one graph, we get:



From the graph, we can see that the point of intersection is $(1, 3)$. Substituting $x = 1$ and $y = 3$ into the two equations, we see:

$$\begin{array}{rrcll} \text{(i)} & 1+3-4 & = & 0 & \text{(true)} \\ \text{(ii)} & 1-3+2 & = & 0 & \text{(true)} \end{array}$$

Hence the point $(1, 3)$ satisfies both equations. Sometimes the point of intersection is not easy to read off the graph, so solving a system of equations algebraically is often easier and more precise.

Example 1 : Solve the system:

(i) $x + y = 4$

(ii) $x - y = -2$

If we add equation (i) to equation (ii) the y's will cancel:

$$\begin{array}{lrcl} \text{(i) + (ii)} & 2x & = & 2 \\ & x & = & 1 \end{array}$$

We now substitute $x = 1$ into either equation (i) or (ii). Let us choose equation (i). Then

$$\begin{array}{rcl} 1+y & = & 3 \\ y & = & 3 \end{array}$$

So the solution is $x = 1$ and $y = 3$. (You can, and should, check the solution by substituting the values of x and y into both equations (i) and (ii).)

Example 2 : Solve the system

$$\begin{array}{rrcl} \text{(i)} & 3x+2y & = & 5 \\ \text{(ii)} & x+2y & = & -3 \end{array}$$

There are the same number of y's i equation (i) and (ii), so if we subtract the equations we would eliminate the y's.

$$\begin{array}{lrcll} \text{(i) - (ii)} & 2x & = & 8 & \text{(Note } 5-(-3)=8\text{)} \\ & x & = & 4 & \end{array}$$

Substitute $x = 4$ in equation (ii):

$$\begin{aligned} 4 + 2y &= -3 \\ 2y &= -7 \\ y &= -3\frac{1}{2} \end{aligned}$$

The solution is $x = 4$, $y = -3\frac{1}{2}$; check that the solution satisfies (i) and (ii).

Example 3 : Solve the system

$$\begin{aligned} (1) \qquad y &= 2x + 3 \\ (2) \qquad y &= 2x + 5 \end{aligned}$$

Subtracting (2) from (1) gives

$$(3) \qquad 0 = -2$$

This is nonsense, and a check shows that there is no solution to this system because the lines that the equations represent are parallel.

Example 4 : Solve the system

$$\begin{aligned} (1) \qquad 2x + 3y &= 10 \\ (2) \qquad 5x + 4y &= 11 \end{aligned}$$

Sometimes it is necessary to multiply both equations by different numbers to get the same multiple of one of the variables. Here is another example of the usefulness of being able to find the lowest common multiple of two numbers. We will take equation (1) times 5, and equation (2) times 2:

$$\begin{aligned} (3) \qquad 10x + 15y &= 50 \\ (4) \qquad 10x + 8y &= 22 \end{aligned}$$

Now subtracting (4) from (3):

$$\begin{aligned} 10x + 15y - (10x + 8y) &= 50 - 22 \\ 7y &= 28 \end{aligned}$$

which gives $y = 4$. Substituting this back into equation (1) gives $x = -1$. A check reveals that $x = -1$ and $y = 4$ is indeed a solution to the original equations.

Exercises:

1. Solve graphically the system of equations

$$x + y = 3$$
$$x - y = 1$$

2. Solve the following systems algebraically where possible.

   (a) $$2x - y = 5$$ $$3x + y = 10$$

   (b) $$2x + 3y = -1$$ $$2x + y = 4$$

   (c) $$3x + 2y = 4$$ $$x + y = 5$$

   (d) $$2x + 5y = -4$$ $$3x + 2y = -6$$

Exercises for Worksheet 3.5

1. How many solutions (one, none, or infinite) will each of the following pairs of equations have?

   (a) $$\begin{aligned} 2x + 3y &= 5 \\ 2x + 3y &= 9 \end{aligned}$$

   (b) $$\begin{aligned} x + 2y &= 3 \\ x - 2y &= 3 \end{aligned}$$

   (c) $$\begin{aligned} x - 2y &= -1 \\ -2x + 4y &= 2 \end{aligned}$$

   (d) $$\begin{aligned} x + 5 &= 3y \\ 4x + y &= 2 \end{aligned}$$

   (e) $$\begin{aligned} y &= \tfrac{1}{3}x + 9 \\ x - 3y &= 2 \end{aligned}$$

2. Solve the following systems of equations.

   (a) $$\begin{aligned} y &= 2x - 3 \\ y &= x + 5 \end{aligned}$$

   (b) $$\begin{aligned} 2x - y &= 4 \\ x + 2y &= 3 \end{aligned}$$

   (c) $$\begin{aligned} 2x - 3y &= 1 \\ 3x + 4y &= -1 \end{aligned}$$

3. The sum of Peter and Anneka's ages is 24, and the difference between their ages is 6. Find their ages given that Peter is older than Anneka.

Worksheet 3.6 Arithmetic and Geometric Progressions

Section 1 Arithmetic Progression

An arithmetic progression is a list of numbers where the difference between successive numbers is constant. The terms in an arithmetic progression are usually denoted as u_1, u_2, u_3 etc. where u_1 is the initial term in the progression, u_2 is the second term, and so on; u_n is the nth term. An example of an arithmetic progression is

$$2, 4, 6, 8, 10, 12, 14, \ldots$$

Since the difference between successive terms is constant, we have

$$u_3 - u_2 = u_2 - u_1$$

and in general

$$u_{n+1} - u_n = u_2 - u_1$$

We will denote the difference $u_2 - u_1$ as d, which is a common notation.

Example 1 : Given that 3,7 and 11 are the first three terms in an arithmetic progression, what is d?

$$7 - 3 = 11 - 7 = 4$$

Then $d = 4$. That is, the common difference between the terms is 4.

If we know the first term in an arithmetic progression , and the difference between terms, then we can work out the nth term, i.e. we can work out what any term will be. The formula which tells us what the nth term in an arithmetic progression is

$$u_n = a + (n - 1) \times d$$

where a is the first term.

Example 2 : If the first 3 terms in an arithmetic progression are 3,7,11 then what is the 10th term? The first term is $a = 3$, and the common difference is $d = 4$.

$$\begin{aligned} u_n &= a + (n-1)d \\ u_{10} &= 3 + (10-1)4 \\ &= 3 + 9 \times 4 \\ &= 39 \end{aligned}$$

Example 3 : If the first 3 terms in an arithmetic progression are 8,5,2 then what is the 16th term? In this progression $a = 8$ and $d = -3$.

$$\begin{aligned} u_n &= a + (n-1)d \\ u_{16} &= 8 + (10-1) \times (-3) \\ &= -37 \end{aligned}$$

Example 4 : Given that $2x, 5$ and $6 - x$ are the first three terms in an arithmetic progression , what is d?

$$\begin{aligned} 5 - 2x &= (6 - x) - 5 \\ x &= 4 \end{aligned}$$

Since $x = 4$, the terms are 8, 5, 2 and the difference is -3. The next term in the arithmetic progression will be -1.

An arithmetic series is an arithmetic progression with plus signs between the terms instead of commas. We can find the sum of the first n terms, which we will denote by S_n, using another formula:

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Example 5 : If the first 3 terms in an arithmetic progression are 3,7,11 then what is the sum of the first 10 terms?
Note that $a = 3$, $d = 4$ and $n = 10$.

$$\begin{aligned} S_{10} &= \frac{10}{2}(2 \times 3 + (10-1) \times 4) \\ &= 5(6 + 36) \\ &= 210 \end{aligned}$$

Alternatively, but more tediously, we add the first 10 terms together:

$$S_{10} = 3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39 = 210$$

This method would have drawbacks if we had to add 100 terms together!

Example 6 : If the first 3 terms in an arithmetic progression are 8,5,2 then what is the sum of the first 16 terms?

$$\begin{aligned} S_{16} &= \frac{16}{2}(2 \times 8 + (16-1) \times (-3)) \\ &= 8(16 - 45) \\ &= -232 \end{aligned}$$

Exercises:

1. For each of the following arithmetic progressions, find the values of a, d, and the u_n indicated.

   (a) 1, 4, 7, ..., (u_{10})

   (b) −8, −6, −4, ..., (u_{12})

   (c) 8, 4, 0, ..., (u_{20})

   (d) −20, −15, −10, ..., (u_6)

   (e) 40, 30, 20, ..., (u_{18})

   (f) −6, −8, −10, ..., (u_{12})

   (g) 2, $2\frac{1}{2}$, 3, ..., (u_{19})

   (h) 6, $5\frac{3}{4}$, $5\frac{1}{2}$, ..., (u_{10})

   (i) −7, $-6\frac{1}{2}$, −6, ..., (u_{14})

   (j) 0, −5, −10, ..., (u_{15})

2. For each of the following arithmetic progressions, find the values of a, d, and the S_n indicated.

   (a) 1, 3, 5, ..., (S_8)

   (b) 2, 5, 8, ..., (S_{10})

   (c) 10, 7, 4, ..., (S_{20})

   (d) 6, $6\frac{1}{2}$, 7, ..., (S_8)

   (e) −8, −7, −6, ..., (S_{14})

   (f) −2, 0, 2, ..., (S_5)

   (g) −20, −16, −12, ..., (S_4)

   (h) 40, 35, 30, ..., (S_{11})

   (i) 12, $10\frac{1}{2}$, 9, ..., (S_9)

   (j) −8, −5, −2, ..., (S_{20})

Section 2 Geometric Progressions

A geometric progression is a list of terms as in an arithmetic progression but in this case the ratio of successive terms is a constant. In other words, each term is a constant times the term that immediately precedes it. Let's write the terms in a geometric progression as u_1, u_2, u_3, u_4 and so on. An example of a geometric progression is

$$10, 100, 1000, 10000, \ldots$$

Since the ratio of successive terms is constant, we have

$$\begin{aligned} \frac{u_3}{u_2} &= \frac{u_2}{u_1} \quad \text{and} \\ \frac{u_{n+1}}{u_n} &= \frac{u_2}{u_1} \end{aligned}$$

The ratio of successive terms is usually denoted by r and the first term again is usually written a.

Example 1 : Find r for the geometric progression whose first three terms are 2, 4, 8.

$$\frac{4}{2} = \frac{8}{4} = 2$$

Then $r = 2$.

Example 2 : Find r for the geometric progression whose first three terms are 5, $\frac{1}{2}$, and $\frac{1}{20}$.

$$\frac{1}{2} \div 5 = \frac{1}{20} \div \frac{1}{2} = \frac{1}{10}$$

Then $r = \frac{1}{10}$.

If we know the first term in a geometric progression and the ratio between successive terms, then we can work out the value of any term in the geometric progression . The nth term is given by

$$\boxed{u_n = ar^{n-1}}$$

Again, a is the first term and r is the ratio. Remember that $ar^{n-1} \neq (ar)^{n-1}$.

Example 3 : Given the first two terms in a geometric progression as 2 and 4, what is the 10th term?

$$a = 2 \qquad r = \frac{4}{2} = 2$$

Then $u_{10} = 2 \times 2^9 = 1024$.

Example 4 : Given the first two terms in a geometric progression as 5 and $\frac{1}{2}$, what is the 7th term?

$$a = 5 \qquad r = \frac{1}{10}$$

Then

$$\begin{aligned} u_7 &= 5 \times (\frac{1}{10})^{7-1} \\ &= \frac{5}{1000000} \\ &= 0.000005 \end{aligned}$$

A geometric series is a geometric progression with plus signs between the terms instead of commas. So an example of a geometric series is

$$1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$$

We can take the sum of the first n terms of a geometric series and this is denoted by S_n:

$$\boxed{S_n=\frac{a(1-r^n)}{1-r}}$$

Example 5 : Given the first two terms of a geometric progression as 2 and 4, what is the sum of the first 10 terms? We know that $a=2$ and $r=2$. Then

$$\begin{aligned} S_{10} &= \frac{2(1-2^{10})}{1-2} \\ &= 2046 \end{aligned}$$

Example 6 : Given the first two terms of a geometric progression as 5 and $\frac{1}{2}$, what is the sum of the first 7 terms? We know that $a=5$ and $r=\frac{1}{10}$. Then

$$\begin{aligned} S_7 &= \frac{5(1-\frac{1}{10}^7)}{1-\frac{1}{10}} \\ &= 5\frac{1-\frac{1}{10^7}}{\frac{9}{10}} \\ &= 5.555555 \end{aligned}$$

In certain cases, the sum of the terms in a geometric progression has a limit (note that this is summing together an infinite number of terms). A series like this has a limit partly because each successive term we are adding is smaller and smaller (but this fact in itself is not enough to say that the limiting sum exists). When the sum of a geometric series has a limit we say that S_∞ exists and we can find the limit of the sum. For more information on limits, see worksheet 3.7. The condition that S_∞ exists is that r is greater than -1 but less than 1, i.e. $|r|<1$. If this is the case, then we can use the formula for S_n above and let n grow arbitrarily big so that r^n becomes as close as we like to zero. Then

$$\boxed{S_\infty=\tfrac{a}{1-r}}$$

is the limit of the geometric progression so long as $-1<r<1$.

Example 7 : The geometric progression whose first two terms are 2 and 4 does not have a S_∞ because $r = 2 \not< 1$.

Example 8 : For the geometric progression whose first two terms are 5 and $\frac{1}{2}$, find S_∞. Note that $r = \frac{1}{10}$ so $|r| < 1$, so that S_∞ exists. Now

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{5}{1-\frac{1}{10}} \\ &= 5\frac{5}{9} \end{aligned}$$

So the sum of $5 + \frac{1}{2} + \frac{1}{20} + \frac{1}{200} + \ldots$ is $5\frac{5}{9}$

Example 9 : Consider a geometric progression whose first three terms are 12, -6 and 3. Notice that $r = -\frac{1}{2}$. Find both S_8 and S_∞.

$$\begin{aligned} S_8 &= \frac{a(1-r^n)}{1-r} \\ &= \frac{12(1-(-\frac{1}{2})^8)}{1-(-\frac{1}{2})} \\ &\approx 7.967 \end{aligned}$$

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{12}{1-(-\frac{1}{2})} \\ &= \frac{12}{3/2} \\ &= 8 \end{aligned}$$

Exercises:

1. Find the term indicated for each of the geometric progressions.

   (a) 1, 3, 9, ..., (u_9)
   (b) 4, -8, 16, ..., (u_{10})
   (c) 18, -6, 2, ..., (u_{12})
   (d) 1000, 100, 10, ..., (u_7)
   (e) 32, -8, 2, ..., (u_{14})
   (f) -0.005, -0.05, -0.5, ..., (u_{10})
   (g) -6, -12, -24, ..., (u_6)
   (h) 1.4, 0.7, 0.35, ..., (u_5)
   (i) 68, -34, 17, ..., (u_9)
   (j) 8, 2, $\frac{1}{2}$, ..., (u_{11})

2. Find the sum indicated for each of the following geometric series

   (a) $6 + 9 + 13.5 + \cdots \ (S_{10})$
   (b) $18 - 9 + 4.5 + \cdots \ (S_{12})$
   (c) $6 + 3 + \frac{3}{2} + \cdots \ (S_{10})$
   (d) $6000 + 600 + 60 + \cdots \ (S_{20})$
   (e) $80 - 20 + 5 + \cdots \ (S_9)$

Exercises for Worksheet 3.6

1. For each of the following progressions, determine whether it is arithmetic, geometric, or neither:

   (a) 5, 9, 13, 17, ...

   (b) 1, −2, 4, −8, ...

   (c) 1, 1, 2, 3, 5, 8, 13, 21, ...

   (d) 81, −9, 3, $\frac{1}{3}$, ...

   (e) 512, 474, 436, 398, ...

2. Find the sixth and twentieth terms, and the sum of the first 10 terms of each of the following sequences:

   (a) −15, −9, −3, ...

   (b) $\log 7$, $\log 14$, $\log 28$, ...

   (c) $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, ...

   (d) 0.5, 0.45, 0.405, ...

   (e) 64,−32,16, ...

3. (a) The third and eighth terms of an AP are 470 and 380 respectively. Find the first term and the common difference. Hint: write expressions for u_3 and u_8 and solve simultaneously.

   (b) Find the sum to 5 terms of the geometric progression whose first term is 54 and fourth term is 2.

   (c) Find the second term of a geometric progression whose third term is $\frac{9}{4}$ and sixth term is $-\frac{16}{81}$.

   (d) Find the sum to n terms of an arithmetic progression whose fourth and fifth terms are 13 and 15.

4. (a) A university lecturer has an annual salary of \$40,000. If this increases by 2% each year, how much will she have grossed in total after 10 years?

   (b) A bob of a pendulum swings through an arc of 50 cm on its first swing. Each successive swing is 90% of the length of the previous swing. Find the total distance the bob travels before coming to rest.

Worksheet 3.7 Continuity and Limits

Section 1 Limits

Limits were mentioned without very much explanation in the previous worksheet. We will now take a closer look at limits and, in particular, the limits of functions. Limits are very important in maths, but more specifically in calculus.

To begin with, we will look at two geometric progressions:

1. $2, 4, 8, 16, \ldots$
2. $5, \frac{1}{2}, \frac{1}{20}, \frac{1}{200}, \ldots$

In the first geometric progression, successive terms get larger and larger as we go along the list. Recall from the last worksheet that the nth term for this geometric progression is

$$u_n = 2 \times 2^{n-1}$$

As n increases, u_n gets larger and larger, and we can make u_n as large as we wish by taking a suitable value for n. The second geometric progression also has infinitely many terms, but in this case the terms are getting smaller and smaller as the list goes on. The nth term for this geometric progression is

$$u_n = 5(\frac{1}{10})^{n-1}$$

So u_{10}, say, is

$$\begin{aligned} u_{10} &= 5(\frac{1}{10})^{10-1} \\ &= \frac{5}{10^9} \\ &= 0.000000005 \end{aligned}$$

which is very small. Indeed, as we take n larger and larger, the terms seem to be getting nearer to zero. We can make the nth term as close as we like to zero by taking a suitably large n. Note that, even though the terms are getting nearer to zero, they will never actually equal zero, no matter how large we make n. But what we do say is that the limit of the geometric progression is zero and we write this as

$$\lim_{n\to\infty} \frac{5}{10^{n-1}} = 0$$

We read this statement as follows: the limit as n tends to ∞ of $\frac{5}{10^{n-1}}$ is zero. In the first geometric progression that we looked at, where the terms got bigger and bigger as n increased, we say that that geometric progression has no limit.

We can find the limit of a function $f(x)$ as $x \to \infty$. For a given function, we will look at what happens as x takes on larger and larger values and work out a general trend. Let's look at the function

$$f(x) = \frac{1}{x}$$

For large values of x, $f(x)$ is very small. As x gets larger, $f(x) \to 0$, so we can say that

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

even though there is no x such that $f(x) = 0$. Now we investigate $f(x) = \frac{1}{x}$ as $x \to 0$. So we are trying to find

$$\lim_{x\to 0} \frac{1}{x}$$

We takes values of x closer and closer to zero and see what happens.

$$\begin{aligned} f(1) &= 1 \\ f(0.01) &= 100 \\ f(0.00001) &= 100000 \end{aligned}$$

It appears that $f(x)$ is getting bigger and bigger as $x \to 0$. Therefore

$$\lim_{x\to 0} \frac{1}{x} \quad \text{does not exist}$$

Notice that $f(x) = \frac{1}{x}$ is not actually defined at $x = 0$ as we are not allowed to divide by zero. Instead of approaching zero by starting at 1 and getting smaller, what happens if we start at $x = -1$ and approach zero from there?

$$\begin{aligned} f(-1) &= -1 \\ f(-0.01) &= -100 \\ f(-0.00001) &= -100,000 \end{aligned}$$

Again, the limit does not exist, but now $f(x)$ gets further away from zero in the negative direction as x gets closer to zero. When looking at the limit of a function as it tends to some finite value, it is important to check values of x on both sides of the value you are looking at.

<u>Example 1</u> : If $f(x) = 1 - 2x$ find

$$\lim_{x\to 0} f(x)$$

We try a few values:

$$\begin{aligned} f(0.01) &= .98 \\ f(-0.01) &= 1.02 \\ f(0.0001) &= .9998 \\ f(-0.0001) &= 1.0002 \end{aligned}$$

These seem to be getting closer to 1 as $x \to 0$, and if we evaluate the function at $x = 0$ we get $f(0) = 1$. This is a convenient check, but remember that limits are actually looking at what happens to a function as x approaches a certain point.

Example 2 : Evaluate $\lim_{x\to 3}(5x + 2)$. It is easy to see that as $x \to 3$ (from both directions) that $f(x)$ approaches 17.

Example 3 : Evaluate

$$\lim_{x\to 5} \frac{x^2 - 25}{x - 5}$$

The function as written is not defined at $x = 5$ since putting $x = 5$ would give us a zero denominator, but we can factorize the numerator to give

$$\begin{aligned} \lim_{x\to 5} \frac{x^2 - 25}{x - 5} &= \lim_{x\to 5} \frac{(x + 5)(x - 5)}{x - 5} \\ &= \lim_{x\to 5}(x + 5) \\ &= 10 \end{aligned}$$

The step where we divide the numerator and denominator by $x - 5$ is only valid for $x \neq 5$, but the point of limits is to look at what is happening close to 5, not actually at 5.

Example 4 :

$$\begin{aligned} \lim_{x\to 0} \frac{x^3 + 2x^2}{x^2} &= \lim_{x\to 0} \frac{x^2(x + 2)}{x^2} \\ &= \lim_{x\to 0} x + 2 \\ &= 2 \end{aligned}$$

Sometimes we are asked to find the limit as $x \to \infty$ of a function, and it is unclear what the limit is. For instance, in the case of

$$f(x) = \frac{3x^2 + x + 2}{2x^2 + 3x + 1}$$

we can't cancel any factors or simplify the expression. Since both the numerator and denominator get large as x gets large, it is not clear whether $f(x)$ gets large or not. We now discuss two methods that we can use to find the limit in cases like this.

Method A

We can divide the numerator and the denominator by the highest power of x in the denominator.

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{3x^2+x+2}{2x^2+3x+1} \\ &= \lim_{x\to\infty} \frac{3+\frac{1}{x}+\frac{2}{x^2}}{2+\frac{3}{x}+\frac{1}{x^2}} \\ &= \frac{3}{2}\end{aligned}$$

The first step is to divide every term in both the numerator and the denominator by x^2. The second, and last, step follows because all terms except the 3 on top and the 2 on the bottom approach 0 as x approaches ∞.

Method B

Recall that as $x \to \infty$ then $\frac{1}{x} \to 0$, and as $x \to 0$ then $\frac{1}{x} \to \infty$. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0} f(\frac{1}{x})$$

To find the limit as $x \to \infty$ of $f(x)$, we can equivalently look at $x \to 0$ of $f(\frac{1}{x})$.

$$\begin{aligned}\lim_{x\to\infty} \frac{3x^2+x+2}{2x^2+3x+1} &= \lim_{x\to 0} \frac{3\frac{1}{x^2}+\frac{1}{x}+2}{2\frac{1}{x^2}+3\frac{1}{x}+1} \\ &= \lim_{x\to 0} \frac{\frac{3+x+2x^2}{x^2}}{\frac{2+3x+x^2}{x^2}} \\ &= \frac{3}{2}\end{aligned}$$

Exercises:

1. Find the following limits, if they exist.

   (a) $\lim_{x\to 3}(4x+1)$

   (b) $\lim_{x\to 2} \frac{x^2-4}{x-2}$

   (c) $\lim_{x\to\infty} \frac{4x^2-2x+7}{3x^2+6x-5}$

   (d) $\lim_{x\to 0} \frac{6}{x}$

   (e) $\lim_{x\to -3} \frac{x^2+5x+6}{2x+6}$

   (f) $\lim_{x\to 1} x^2+3$

(g) $\lim_{x \to 0} \frac{x+2}{x^2}$

(h) $\lim_{x \to 6} \frac{3x-18}{x^2-36}$

(i) $\lim_{x \to \infty} 2x+1$

(j) $\lim_{x \to \infty} \frac{1}{x} - 4$

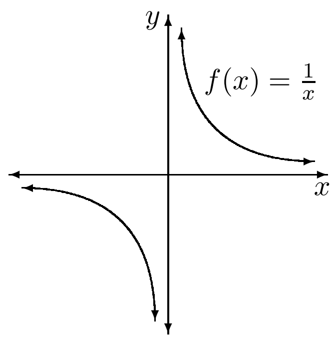
Section 2 Continuity

Limits help to sketch the graphs of functions on the $x - y$ plane. They tell how the function behaves as it gets close to certain values of x and what value the function tends to as x gets large, both positively and negatively.

If the limit of a function does not exist at a certain finite value of x, then the function is discontinuous at that point.

> Example 1 : Given that $f(x) = \frac{1}{x}$, we know that $\lim_{x\to\infty} f(x) = 0$ and that $\lim_{x\to 0} f(x)$ does not exist. The function $f(x)$ is not defined at $x = 0$.

The graph of $y = f(x)$ is drawn below:



For a function to be continuous at $x = c$ we need three conditions to be met.

1. $f(x)$ must be defined at $x = c$
2. $\lim_{x\to c} f(x)$ must exist
3. $f(c)$ must equal $\lim_{x\to c} f(x)$

Note that $f(x) = \frac{1}{x}$ is not continuous at $x = 0$ because $f(0)$ is not defined; neither does $\lim_{x\to 0} f(x)$ exist.

When asked to test for continuity, the first thing that we check for is whether or not the function is defined at the point in question. Then we can check the limit of the function as x tends to that value. Finally, we would check that $f(c) = \lim_{x\to c} f(x)$.

Example 2 : We define $f(x)$ as

$$f(x) = \begin{cases} \frac{x^2+9}{2} & \text{when } x \neq 0 \\ 5 & \text{when } x = 0 \end{cases}$$

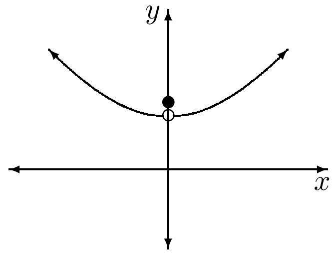
Is the function continuous at $x = 0$? The function is defined at zero: $f(0) = 5$.

$$\lim_{x\to 0} \frac{x^2+9}{2} = 4\frac{1}{2}$$

Notice that we are looking at values close to zero, but not actually zero, so we use the appropriate part of the function. The limit exists as $x \to 0$. But

$$\lim_{x\to 0} \frac{x^2+9}{2} = 4\frac{1}{2} \neq f(0)$$

Therefore the function is not continuous at $x = 0$. The graph of $y = f(x)$ is drawn below:



In general, for simple functions, there is a rule of thumb that says that if you can draw the graph of the function without lifting your pen from the paper, then the function is almost certainly continuous for those values of x.

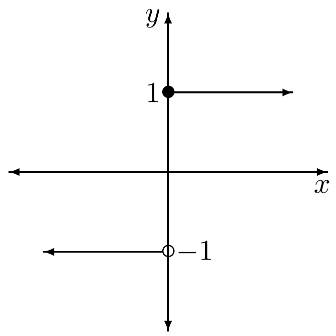
Example 3 : The function $f(x) = \frac{3}{x+1}$ is not continuous at $x = -1$ since it is not defined at $x = -1$.

Example 4 : The function $f(x) = \frac{5x}{x^2+4x+3} = \frac{5x}{(x+3)(x+1)}$ is not continuous at $x = -1$ and $x = -3$ since it is not defined at those values.

Example 5 : The function

$$f(x) = \begin{cases} -1 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases}$$

is defined for all values of x, but $\lim_{x\to 0} f(x)$ does not exist, since if we take values of x close to zero on the negative side we get -1, but if we take values of x close to zero on the positive side we get +1. Therefore $f(x)$ is discontinuous at $x = 0$.



Exercises:

1. Which of the following limits exist? If they exist, evaluate them.

   (a) $\displaystyle\lim_{x\to 2} \frac{3}{x-2}$

   (b) $\displaystyle\lim_{x\to 4} \frac{x+4}{2}$

   (c) $\displaystyle\lim_{x\to 0} \frac{x^2+3x}{2x}$

   (d) $\displaystyle\lim_{x\to \infty} \frac{x^2+6x+8}{5x^2+4}$

   (e) $\displaystyle\lim_{x\to 0} \frac{6}{x}$

Exercises for Worksheet 3.7

1. Evaluate the following:

   (a) $\lim_{x\to 3}(2x+4)$

   (b) $\lim_{x\to 3}\frac{x^2-9}{x-3}$

   (c) $\lim_{x\to 3}\frac{1}{x}$

   (d) $\lim_{x\to 3}\frac{3x^2-2x+4}{4x^2-7}$

   (e) $\lim_{x\to 3}\frac{(2x-6)(x+3)}{\sqrt{x^2-9}}$

2. Which of the following are continuous at $x=1$?

   (a) $f(x)=\begin{cases} 5 & x=1 \\ 2x+3 & x\neq 1 \end{cases}$

   (b) $f(x)=\begin{cases} 4x & x=1 \\ \frac{2x^2-2}{x-1} & x\neq 1 \end{cases}$

   (c) $f(x)=\begin{cases} 0 & x=1 \\ \frac{x-1}{\sqrt{1-x}} & x\neq 1 \end{cases}$

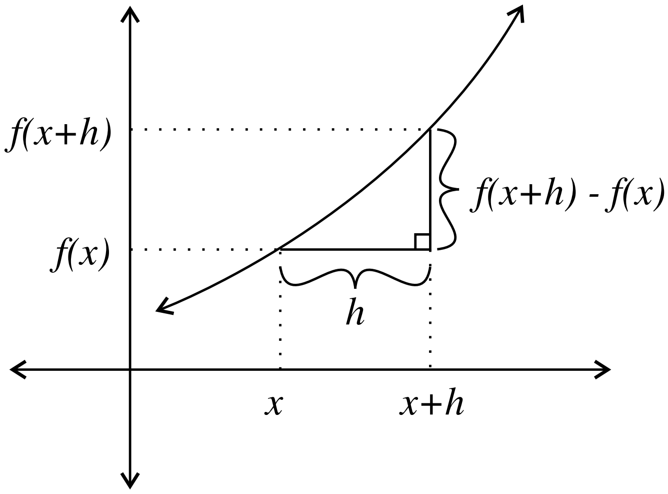
   (d) $f(x)=\begin{cases} \left|2x-3\right| & x=1 \\ 2x-3 & x\neq 1 \end{cases}$

   (e) $f(x)=\begin{cases} x-3 & x\leq 1 \\ x^2-4x+3 & x>1 \end{cases}$

Worksheet 3.8 Introduction to Differentiation

Section 1 Definition of Differentiation

Differentiation is a process of looking at the way a function changes from one point to another. Given any function we may need to find out what it looks like when graphed. Differentiation tells us about the slope (or rise over run, or gradient, depending on the tendencies of your favourite teacher). As an introduction to differentiation we will first look at how the derivative of a function is found and see the connection between the derivative and the slope of the function.



Given the function $f(x)$, we are interested in finding an approximation of the slope of the function at a particular value of x. If we take two points on the graph of the function which are very close to each other and calculate the slope of the line joining them we will be approximating the slope of $f(x)$ between the two points. Our x-values are x and $x+h$, where h is some small number. The y-values corresponding to x and $x+h$ are $f(x)$ and $f(x+h)$. The slope m of the line between the two points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are the two points. In our case, we have the two points $(x, f(x))$ and $(x+h, f(x+h))$. So the slope of the line joining them is given by

$$\begin{aligned} m &= \frac{f(x+h) - f(x)}{x+h-x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

Example 1 : Let $f(x) = x^3$. Find the slope of the line joining $(x, f(x))$ and

$(x+h, f(x+h))$. From our definitions,

$$\begin{aligned} m &= \frac{f(x+h)-f(x)}{h} \\ &= \frac{(x+h)^3 - x^3}{h} \\ &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= 3x^2 + 3xh + h^2 \end{aligned}$$

Example 2 : Let $f(x) = 2x + 5$. Find the slope of the line joining the points $(1, f(1))$ and $(1.01, f(1.01))$.

$$\begin{aligned} m &= \frac{f(1.01)-f(1)}{1.01-1} \\ &= \frac{7.02-7}{0.01} \\ &= \frac{0.02}{0.01} \\ &= 2 \end{aligned}$$

as expected since the gradient of $y = 2x + 5$ is 2.

Example 3 : Let $f(x) = x^2$. Find the slope of the line joining $(x, f(x))$ and $(x+h, f(x+h))$ if $h = 0.1$ and $x = 1$.

$$\begin{aligned} m &= \frac{f(x+h)-f(x)}{h} \\ &= \frac{f(1+0.1)-f(1)}{0.1} \\ &= \frac{f(1.1)-f(1)}{0.1} \\ &= \frac{(1.1)^2-(1)^2}{0.1} \\ &= \frac{0.21}{0.1} \\ &= 2.1 \end{aligned}$$

The smaller that h gets to zero, the closer x and $x+h$ get to each other, and consequently the better m approximates the slope of the function at the point $(x, f(x))$. So we look at what happens when we take the limit as $h \to 0$ in the slope formula and we call this the derivative $f'(x)$. So

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Notice that $f'(x)$ is the derivative only if the limit exists. If the limit does not exist at particular x-values then we say that the function is not differentiable at those x-values.

Example 4 : Find the derivative of $f(x) = x^2 + 3$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 + 3 - (x^2+3)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 3 - x^2 - 3}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= 2x \end{aligned}$$

Note: There are other notations for the derivative of a function in x. The most common are $f'(x)$ and $\frac{df}{dx}$. If $y = f(x)$, we also use $y' = f'(x)$ or $\frac{dy}{dx}$ to refer to the derivative.

Example 5 : Find the derivative of the function $f(x) = 2x + 5$ at $x = 1$.

$$\begin{aligned} f(x+h) &= 2(x+h) + 5 \\ f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{2(1+h) + 5 - 7}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} \\ &= \lim_{h\to 0} 2 \\ &= 2 \end{aligned}$$

Example 6 : Find the derivative of $y = |x|$ at $x = 0$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$
\begin{aligned}
f'(0) &= \lim_{h \to 0} \frac{|0+h| - |0|}{h} \\
&= \lim_{h \to 0} \frac{|h|}{h}
\end{aligned}
$$

Recall that

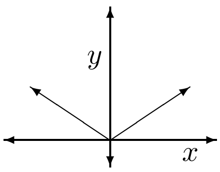
$$
|h| = \begin{cases} h & \text{when } h \geq 0 \\ -h & \text{when } h < 0 \end{cases}
$$

The absolute-value sign prevents us from simply canceling. Let's look at what $\frac{|h|}{h}$ equals. h could be very small and negative, in which case $f'(0) = -1$. Or it could be very small and positive, in which case $f'(0) = 1$. That is

$$
\begin{aligned}
&\text{if } h < 0, \text{ then } f'(0) = -1 \\
&\text{if } h > 0, \text{ then } f'(0) = 1
\end{aligned}
$$

So the limit does not exist as $h \to 0$ since we get a different value for the limit depending upon whether or not we are close to zero on the negative side or the positive side. Therefore the derivative of $f(x) = |x|$ does not exist at $x = 0$.

Look at the graph of $y = |\, x|$.



The pointed part at $x = 0$ shows a rapid and abrupt change of slope. Functions that have sharp points on their graphs do not have derivatives at these points, although they may have a derivative everywhere else. The function $f(x) = |x|$ is not differentiable at $x = 0$, although it is continuous there.

Exercises:

1. Using the method outlined above, find $f'(x)$ for each of the following functions. That is, use

$$
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
$$

   (a) $f(x) = x^2 + 2$
   (b) $f(x) = 3x - 5$
   (c) $f(x) = 3 - x^2$
   (d) $f(x) = 4x + 5$
   (e) $f(x) = 2 - x$

Section 2 Polynomial Differentiation

Having looked at the general way of finding the derivative of a function, we can now look at those functions for which we already have derivatives and give some simple rules. From these we will be able to determine the derivatives of similar functions. Notice that if we take $f(x) = c$, where c is a constant, we get

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= 0 \end{aligned}$$

The last line is true as $\frac{0}{h} = 0$ for any h except $h = 0$ and limits are about what happens as h gets closer and closer to zero, without actually reaching zero. So for $f(x) = c$ we have $f'(x) = 0$. This is logical since a line such as $y = 2$ which is parallel to the x-axis has a slope of zero.

Now consider $f(x) = ax$. Then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{ax + ah - ax}{h} \\ &= \lim_{h \to 0} \frac{ah}{h} \\ &= a \end{aligned}$$

So if $f(x) = ax$ we get $f'(x) = a$ for any x. Thinking back to worksheet 2.10 where we looked at the function $y = mx + b$, we found that m, the coefficient of x, is the slope of the line. So it makes sense that the derivative of $f(x) = ax$ is a.

Now consider $f(x) = bx^2$. Then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{b(x+h)^2 - bx^2}{h} \\ &= \lim_{h \to 0} \frac{2bxh + bh^2}{h} \\ &= \lim_{h \to 0} (2bx + bh) \\ &= 2bx \end{aligned}$$

So if $f(x) = bx^2$ we get $f'(x) = 2bx$ for any x. We could carry on for higher powers of x and notice the pattern that if

$$\begin{aligned} f(x) &= cx^n \\ \text{then} \quad f'(x) &= ncx^{n-1} \end{aligned}$$

Furthermore, if we have a sum of functions, it can be shown that the derivative of the sum is the sum of the derivatives. This means that if $f(x)$ is a sum of terms that each look like cx^n (in other words, a polynomial) you can use the above rule for each term to determine the derivative of the function.

Example 1 : Let $g(x) = x^2 + 3x + 2$. Then

$$\begin{aligned} g'(x) &= 2x + 3 + 0 \\ &= 2x + 3 \end{aligned}$$

Example 2 : Find the derivative of $f(x) = 5x^3 + 3x^2 + 2^2$.

$$\begin{aligned} f'(x) &= 5 \times 3x^2 + 3 \times 2x^1 + 0 \\ &= 15x^2 + 6x \end{aligned}$$

Example 3 : Find the derivative of $h(x) = 1 + \frac{1}{x} = 1 + x^{-1}$.

$$\begin{aligned} h'(x) &= 0 + -1 \times x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

Example 4 : Given $f(x) = 6x^2 - 4x + 1$, find $f'(2)$.
First find $f'(x)$, then find $f'(2)$.

$$\begin{aligned} f'(x) &= 12x - 4 \\ f'(2) &= 12 \times 2 - 4 \\ &= 20 \end{aligned}$$

Example 5 : Given $f(x) = x^3 - 2x^2 + 5$, find $f'(-1)$.
First find $f'(x)$, then find $f'(-1)$.

$$\begin{aligned} f'(x) &= 3x^2 - 4x \\ f'(-1) &= 3(-1)^2 - 4(-1) \\ &= 7 \end{aligned}$$

Exercises:

1. Find the derivative of each of the following functions

   (a) x^2

   (b) $3x^2 + 4x$

   (c) $x^3 - 6x$

   (d) $6x^2 - 2x + 3$

   (e) $\frac{1}{x} + 4x$

   (f) $3x^2 - x + 2$

   (g) $x^5 + 4x^3 - 7x$

   (h) $\frac{4}{x} - x^2$

   (i) $4x^5 + 6x^3$

   (j) $x^7 + 4x^5$

2. (a) If $f(x) = 2x^3 - 4x$, find $f'(3)$.

   (b) If $f(x) = 7x^2 - 2$, find $f'(-4)$.

   (c) If $f(x) = 5 - 3x^2$, find $f'(1)$.

   (d) If $f(x) = 6x + 7$, find $f'(12)$.

   (e) If $f(x) = 4x^3 - 2x^2 + 4x$, find $f'(5)$.

Section 3 STATIONARY POINTS

Recall that differentiation tells us about the slope of a function at any point on the graph where the function is defined. If $f'(5) = 3$ then, for the function f, we know that the slope of the function at $x = 5$ is 3. If we know the equation of a function, then we could evaluate the slope at various x-values. There are particular points on a graph which are of special interest. These are called stationary points. At a stationary point, the gradient of the function is zero. The stationary points are of interest to us because they help us to draw the graph of a function. There are three different types of stationary points:

1. Minimum points

2. Maximum points

3. Horizontal points of inflection

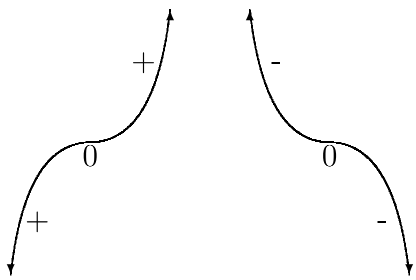
All types of stationary points have the property that the derivative is zero.

Minimum points occur when the graph reaches a local minimum, and has a shape like this:

The gradient changes from negative to zero to positive. We call this concave up, because the cup opens upwards. A maximum occurs when the graph looks like this:

We call this concave down. The gradient changes from positive to zero to negative.

The third type of stationary point, a horizontal point of inflection, occurs when the concavity changes from up to down or from down to up. They look like:



The slope of a horizontal point of inflection momentarily goes to zero where the curve changes concavity. On either side of the point of inflection, the gradient has the same sign, i.e. if the gradient is negative on one side of the point of inflection, then it is negative on the other side. Conversely, for a stationary point that is either a minimum or a maximum, the gradient is negative on one side of the point and positive on the other.

Using this information, we can determine what types of stationary points occur on a graph.

Example 1 : Find the slope of the function $f(x) = x^3 + 3$ at $x = 1$ and $x = 0$.

$$\begin{aligned} f'(x) &= 3x^2 + 0 \\ &= 3x^2 \end{aligned}$$

The slope of the function f at $x = 1$ is the value of the derivative at $x = 1$. So $f'(1) = 3(1)^2 = 3$. The slope of the function $f(x) = x^3 + 3$ at $x = 1$ is 3.

The slope at $x = 0$ is found in the same way. $f'(0) = 3(0)^2 = 0$. The slope at $x = 0$ is 0 so there must be a stationary point at $x = 0$.

Example 2 : Find all the stationary points of $g(x) = x^2 + 2x + 2$.

We have $g'(x) = 2x + 2$. Stationary points occur when $g' = 0$. So we must find all x for which this is true, i.e. we need to solve the equation

$$2x + 2 = 0$$

The only solution for this is $x = -1$, so $x = -1$ is the x-value of the stationary point. To find the other coordinate, we put $x = -1$ in the original function:

$$g(-1) = (-1)^2 + 2(-1) + 2 = 1$$

The only stationary point is $(-1, 1)$. To see what kind of stationary point it is we need to see what the slope is on either side of the stationary point. Now, $-1 + h$ is on the right side of -1 for h small and positive, and on the left side of -1 for h small and negative. The slope at $-1 + h$ is

$$\begin{aligned} g'(-1+h) &= 2(-1+h) + 2 \\ &= -2 + 2h + 2 \\ &= 2h \end{aligned}$$

This is positive for h positive, and negative for h negative. This means that the stationary point is a local minimum.

Example 3 : Find the stationary points of the function $f(x) = x^3 + 3x^2 + 5$.

We calculate $f'(x)$ and find all the x-values that satisfy $f'(x) = 0$.

$$f'(x) = 3x^2 + 6x = 3x(x + 2) = 0$$

This equation has the solutions $x = 0$ and $x = -2$. And $f(0) = 5$, $f(-2) = 9$. So there are two stationary points: $(0, 5)$ and $(-2, 9)$.

Remember that the derivative at any value of x gives you the slope of the function at that value of x.

Example 4 : Given $f(x) = 3x^2 + 2x + 1$, find $f'(-3)$.

First find $f'(x)$, then find $f'(-3)$.

$$\begin{aligned} f'(x) &= 6x + 2 \\ f'(-1) &= 6(-3) + 2) \\ &= -16 \end{aligned}$$

Example 5 : Given $f(x) = 4x^2 - 5x + 7$, find the value of x for which $f'(x) = 11$.
First find $f'(x)$, then solve $f'(x) = 11$.

$$\begin{aligned} f'(x) &= 8x - 5 \\ 11 &= 8x - 5 \\ x &= 2 \end{aligned}$$

When $x = 2$ the slope of the function is 11.

Exercises:

1. Given $f(x) = -2x^2 + 6x - 4$
   (a) find x for which $f'(x) = 20$
   (b) find $f'(2)$
2. Find the stationary points for each of the following functions and state whether they are a maximum, minimum, or a point of inflection.
   (a) $f(x) = x^2 + 6x + 8$
   (b) $f(x) = -x^2 - 2x + 15$
   (c) $f(x) = x^3 + 2$

Section 4 Sketching Graphs

We can use information that we get from derivatives to help sketch graphs of functions. If we can determine the x- and y-intercepts of a function together with the stationary points we can determine roughly what the function looks like. The other bit of information we can find useful is what happens to the function as x approaches positive or negative infinity.

Example 1 : Use the derivatives to help sketch $f(x) = x^3 + 3x^2 + 5$.

As determined in the previous section, $f'(x) = 3x^2 + 6x$, and f has the two stationary points $(0, 5)$ and $(-2, 9)$. Now we should determine the changes of slope on either side of both these points. Look at the x-value $0 + h$, which is to the right of 0 if h is small and positive, and to the left of 0 if h is small and negative. We have $f'(0 + h) = f'(h) = 3(h)^2 + 6(h) = 3h^2 + 6h$. This is positive if h is small and positive, and negative if h is small and negative. The slope is positive to the right and negative to the left. Then $(0, 5)$ is a minimum turning point.

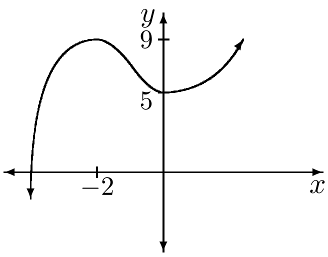
Repeating the process for $x = -2$, we get

$$\begin{aligned} f'(-2+h) &= 3(-2+h)^2 + 6(-2+h) \\ &= -6h + 3h^2 \\ &= 3h(h-2) \end{aligned}$$

This is negative if h is small and positive, but positive if h is small and negative. The slope is positive to the left of -2 , and negative to the right of -2. Then $(-2, 9)$ is a local maximum.

In addition, as $x \to \infty$, $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to -\infty$.

So the graph looks like:



<u>Example 2</u> : Sketch the graph of $f(x) = x^3 + 3$.

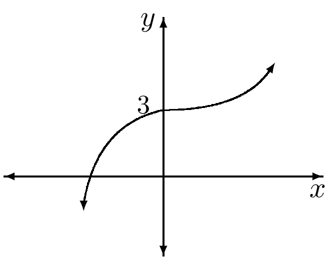
We first solve

$$f'(x) = 3x^2 = 0$$

This has the solution $x = 0$. So the critical point is $(0, 3)$. What happens either side of $x = 0$?

$$f'(0 + h) = 3h^2$$

This is positive for positive or negative h, so we have a horizontal point of inflection. The slope to the left and right of $x = 0$ is positive. The graph looks like:



Exercises:

1. Sketch the following graphs using the process outlined in section 4.

   (a) $f(x) = x^2 + 4x + 3$

   (b) $f(x) = x^3 - 1$

   (c) $f(x) = 2x^3 - 3x^2 - 36x + 18$

   (d) $f(x) = -x^2 - 2x + 15$

   (e) $f(x) = x^2 - 2x - 24$

Exercises for Worksheet 3.8

1. Using the definition

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

evaluate the derivative of the functions given.

   (a) $f(x) = 2x + 3$

   (b) $f(x) = x^2 - 2x + 1$

   (c) $f(x) = x^3$

2. Using the rule $\frac{dx^n}{dx} = nx^{n-1}$, differentiate the following functions with respect to x.

   (a) $x^2 + 6x + 83$

   (b) $7x^3 - 5x^2 + 9x$

   (c) $\sqrt{x} + 8x$

   (d) $3x^{-2} + x^{-1}$

   (e) $\frac{1}{x^2} + \frac{1}{x} + 6x$ (Hint: rewrite $\frac{1}{x^2}$ as x^{-2}.)

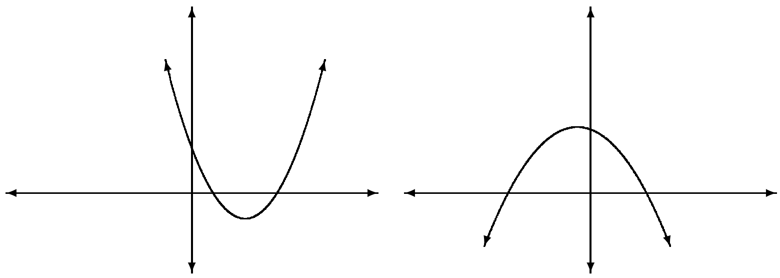
3. (a) Sketch the function $f(x) = 2x^3 - 3x^2 - 12x$, labelling the y-intercepts and the stationary points.

   (b) The air temperature T (degrees Celsius) as a function of height s (kilometres) above sea level is measured by a scientist in a hot-air balloon. The function is given by $T = 20 - 3s$. Find $T'(s)$ and give an interpretation of your answer.

Worksheet 3.9 Further Differentiation

Section 1 Discriminant

Recall that the expression $ax^2 + bx + c$ is called a quadratic, or a polynomial of degree 2. The graph of a quadratic is called a parabola, and looks like one of the following:



They are symmetrical about a stationary point which is either a local minimum or maximum. Parabolas do not have points of inflection. In the quadratic $y = ax^2 + bx + c$, if the co-efficient of x^2 is greater than zero, the parabola is concave up; if a is negative, the parabola is concave down. The option $a = 0$ is precluded as this would result in a linear polynomial which is a straight line when graphed.

The quadratic formula found by solving $ax^2 + bx + c = 0$ is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The value under the square root sign, $b^2 - 4ac$, is called the discriminant, and we denote this by Δ. We can tell quite a lot about the curve $y = ax^2 + bx + c$ just by evaluating $\Delta = b^2 - 4ac$.

If $\Delta > 0$	then the graph will cut the x-axis in two places, i.e. there are two x-intercepts.
If $\Delta = 0$	the graph will touch the x-axis in one place only.
If $\Delta < 0$	the graph will sit wholly above or below the x-axis depending on the sign of a. There are no x-intercepts.

Using this information and the information gained from the derivative, we can sketch the graph of any quadratic.

<u>Example 1</u> : Sketch the graph of $y = x^2 + 3x + 2$. We have $a = 1, b = 3$, and $c = 2$. Since $a > 0$, the stationary point is a minimum. Further, $b^2 - 4ac > 0$ so there are two x-intercepts, which are found in this case through factorization:

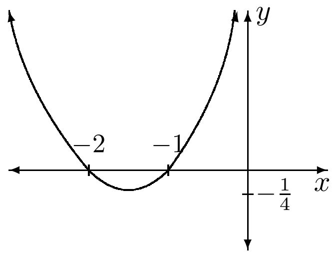
$x^2 + 3x + 2 = (x+2)(x+1)$. So $y = 0$ when $(x+2)(x+1) = 0$, or when $x = -1$ or -2. We find where the stationary point is by setting $\frac{dy}{dx} = 0$. This gives

$$\frac{dy}{dx} = 2x + 3 = 0$$

which has the solution $x = -\frac{3}{2}$. When $x = -\frac{3}{2}$,

$$y = (-\frac{3}{2})^2 + 3(-\frac{3}{2}) + 2 = -\frac{1}{4}$$

The stationary point is at $(-\frac{3}{2}, -\frac{1}{4})$. When $x = 0$, $y = 2$, so the y-intercept is 2. With all this information at our disposal, we can now draw the graph of $y = x^2 + 3x + 2$:



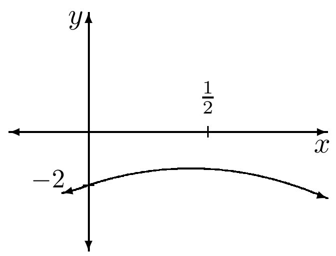
Example 2 : Sketch $y = -2 + x - x^2$. This can be rewritten as $y = -x^2 + x - 2$ so that $a = -1$, $b = 1$, and $c = -2$. We have $a < 0$ so the stationary point is a maximum. We find the x-coordinate of the stationary point by setting $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = -2x + 1 = 0$$

which has the solution $x = \frac{1}{2}$, and the corresponding y value is

$$y = -(\frac{1}{2})^2 + \frac{1}{2} - 2 = -\frac{7}{4}$$

The stationary point has the coordinates $(\frac{1}{2}, -\frac{7}{4})$. The discriminant is $b^2 - 4ac = -7 < 0$ so there are no x-intercepts. The y-intercept is given when $x = 0$, so $y = -2$. The graph of $y = -2 + x - x^2$ then looks like:



Exercises:

1. Sketch the graph of each of the following using the method studied in section 1.

   (a) $y = x^2 + 7x + 12$

   (b) $y = -x^2 + x + 6$

   (c) $y = x^2 + 4x + 5$

   (d) $y = x^2 - 16$

   (e) $y = 2x^2 - 5x - 3$

Section 2 Second Derivatives

Recall from the last worksheet the discussion on concavity. To determine whether a stationary point was a maximum, minimum, or a point of inflection, we looked at the changes in slope as we moved from one side of the stationary point to the other. There is an easier method of determining the concavity of a graph at any point (not just a critical point). The method is based upon the notion that concavity is a measure of the change of a slope.

As we viewed the derivative as a measure in the change of the height of a function we can view the derivative of the derivative as a change in the slope of the graph. The derivative of the derivative is called the second derivative, and it is denoted by one of the following:

$$f''(x),\ y'',\ \frac{d^2y}{dx^2},\ \text{or}\ D^2(f)$$

depending on how the function is defined.

Example 1 : Find the second derivative of $y = 5x^2 + 3$.

$$\begin{aligned} \frac{dy}{dx} &= 10x \\ \frac{d^2y}{dx^2} &= 10 \end{aligned}$$

So the second derivative of $y = 5x^2 + 3$ is 10.

Example 2 : Find the second derivative of $f(x) = x^3 + 3x^2 + 2x$.

$$\begin{aligned} f'(x) &= 3x^2 + 6x + 2 \\ f''(x) &= 6x + 6 \end{aligned}$$

The second derivative is a help with curve sketching as it tells about the concavity of the graph at any point - this is most useful at critical points.

If $f''(x) = 0$, the concavity is changing, so that the critical point is a point of inflection.

If $f''(x) > 0$, then the graph is concave up at x.

If $f''(x) < 0$, then the graph is concave down at x.

This means

(1) If $f'(x) = 0$ and $f''(x) > 0$ there is a minimum turning point at x.

(2) If $f'(x) = 0$ and $f''(x) < 0$ there is a maximum turning point at x.

Example 3 : For the function $f(x) = x^3 + 3x^2 + 3x$, find the stationary points and describe their important characteristics. We first find the solutions of $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 3x^2 + 6x + 3 \\ &= 3(x^2 + 2x + 1) \\ &= 3(x+1)(x+1) \\ &= 0 \end{aligned}$$

which has the solution $x = -1$. Also $f(-1) = (-1)^3 + 3(-1)^2 + 3(-1) = -1$. So the critical point is at $(-1, -1)$. Now,

$$f''(x) = 6x + 6$$

At $x = -1$, $f''(-1) = 6(-1) + 6 = 0$. Therefore the concavity is changing at $x = -1$, so the point $(-1, -1)$ is a point of inflection.

Example 4 : Find the stationary points of the function $f(x) = x^4 - x^2 + 1$, and describe their properties. We first find solutions of $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 4x^3 - 2x \\ &= 2x(2x^2 - 1) \end{aligned}$$

When $f'(x) = 0$, $2x(2x^2 - 1) = 0$ which has the solutions $x = 0$ and $x = \pm\frac{1}{\sqrt{2}}$. These are the x-coordinates of the critical points. The second derivative is $f''(x) = 12x^2 - 2$.

At $x = 0$, $f''(0) = -2 < 0$ so at $x = 0$ we have a local maximum.
At $x = \frac{1}{\sqrt{2}}$, $f''(\frac{1}{\sqrt{2}}) = 4 > 0$ so at $x = 0$ we have a local minimum.
At $x = -\frac{1}{\sqrt{2}}$, $f''(-\frac{1}{\sqrt{2}}) = 4 > 0$ so at $x = 0$ we have a local minimum.

Exercises:

1. Find the stationary points of each of the following and describe their important characteristics.

   (a) $f(x) = x^3 - 12x - 4$

   (b) $f(x) = 2x^4 - x + 6$

   (c) $f(x) = 2x^3 - 4x^2 + 8$

   (d) $f(x) = x^2 - 8x + 7$

   (e) $f(x) = 6x^2 + 4x - 6$

Section 3 Further Sketching

We now have a comprehensive range of tools that help us sketch curves, especially those of polynomial functions. We can find intercepts, examine what happens to the function at certain values of x, find critical points, and find properties of critical points. Let's put these tools to use.

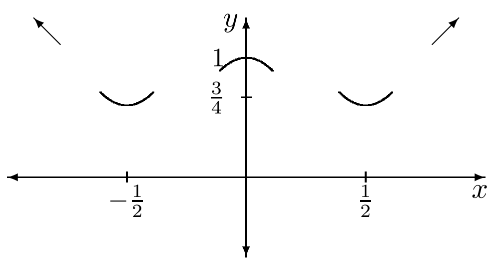
Example 1 : From the information in example 4 in section 2, sketch the function $y = f(x) = x^4 - x^2 + 1$. The critical points were at $x = 0$, and at $x = \pm\frac{1}{\sqrt{2}}$.

At $x = 0$, $f(0) = 1$, and this is a local maximum.

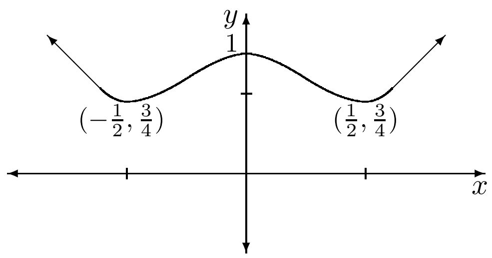
At $x = \frac{1}{\sqrt{2}}$, $f(x) = \frac{3}{4}$ and this point was a local minima.

At $x = -\frac{1}{\sqrt{2}}$, $f(x) = \frac{3}{4}$ and this point was a local minima.

The y-intercept is $(0, 1)$. We leave the question of the x-intercepts for the moment. As $x \to \pm\infty$, $f(x) \to \infty$. From all this information, we can now draw some of $f(x) = x^4 - x^2 + 1$:



Since the three critical points shown are all the critical points, there can be no other changes in direction, so the graph can be completed as follows:



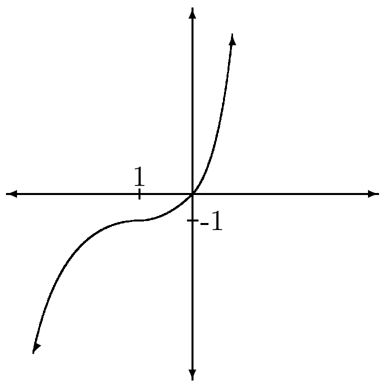
Example 2 : In section 2, example 3, we looked at the function $f(x) = x^3+3x^2+3x$. Sketch this function. There was only one critical point, $(-1, -1)$, and it was a point of inflection. The y-intercept is found by letting $x = 0$, which gives $f(0) = 0$. The x-axis intercepts are given by the solutions to $f(x) = 0$:

$$x^3 + 3x^2 + 3x = x(x^2 + 3x + 3) = 0$$

So one solution is $x = 0$. The solutions to $x^2 + 3x + 3 = 0$ are given by

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{9-12}}{2} \\ &= \frac{-3 \pm \sqrt{-3}}{2} \end{aligned}$$

which has no real solutions. So $x = 0$ is the only x-intercept. As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$. The graph of $f(x) = x^3 + 3x^2 + 3x$ then looks like:



Example 3 : Sketch the function $f(x) = x^3 - 12x + 3$.
$f'(x) = 3x^2 - 12$ which is zero when either $x = 0$ or $x = \pm 2$. $f''(x) = 6x$. We have

$f''(0) = 0$.
$f''(2) = 12 \Rightarrow$ minimum at $(2, -13)$.
$f''(0) = 0 \Rightarrow$ maximum at $(-2, 19)$.

There is an inflection point at $(0, 3)$. The graph of $f(x)$ is shown:

Exercises:

1. Using the method outlined in section 3, sketch the following curves.

   (a) $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 2$

   (b) $f(x) = -x^2 - 2x + 8$

   (c) $f(x) = x^3 - 28x + 48$

   (d) $f(x) = \frac{1}{3}x^3 + 6x^2 + 35$

   (e) $f(x) = x^3 + 6x^2 + 7$

Exercises for Worksheet 3.9

1. For each of the following quadratic functions, find the sign of the discriminant to determine if there are 0,1, or 2 roots.

   (a) $f(x) = x^2 - 5x + 6$

   (b) $f(x) = 4x^2 - x + 2$

   (c) $f(x) = 4x^2 - x - 2$

   (d) $f(x) = 4x^2 - 12x + 9$

   (e) $f(x) = 3x^2 - 5x + 3$

2. For each of the following, sketch a function which satisfies the given conditions.

   (a) $f'(x) > 0$ and $f''(x) > 0$

   (b) $f'(x) > 0$ and $f''(x) < 0$

   (c) $f'(x) < 0$ and $f''(x) > 0$

   (d) $f'(x) < 0$ and $f''(x) < 0$

   (e) $f(2) = 4$, $f'(2) = 0$, $f''(x) < 0$ for $x < 2$, and $f''(x) > 0$ for $x > 2$.

3. (a) Sketch the function $f(x) = x^3 - 3x^2 - 9x + 27$, labelling all intercepts and critical points. Determine the nature of the critical points.

   (b) An economist stated: 'Although our current account deficit is increasing, the government policies to reduce it seem to be taking effect'. If D represents the current account deficit, and t represents time, what can be determined about $\frac{dD}{dt}$ and $\frac{d^2D}{dt^2}$?

Worksheet 3.10 Differentiating Special Functions

Section 1 Exponentials

So far in the worksheets, we have only really covered the differentiation of polynomials. We will need to be able to differentiate other functions as well. This worksheet deals with the rules for differentiating some special functions. How these rules come about will not be shown, as this is a bit complicated for first-year maths.

Recall the properties of the exponential and logarithmic functions:

$$\text{if } \; y = e^x \quad \text{then} \quad x = \log_e y$$

We need to know the derivative of both these functions, which are given by

$$\begin{aligned} \frac{d\,e^x}{dx} &= e^x \\ \text{and} \quad \frac{d\,\log_e x}{dx} &= \frac{1}{x} \end{aligned}$$

These are rules that we need to remember - don't worry about where they come from. They can each be slightly generalized to:

$$\begin{aligned} \frac{d\,e^{g(x)}}{dx} &= g'(x)e^{g(x)} \\ \text{and} \quad \frac{d\,\log_e k(x)}{dx} &= \frac{k'(x)}{k(x)} \end{aligned}$$

Example 1 : Find the derivative of e^{5x^2}. We let $g(x) = 5x^2$. Then $g'(x) = 10x$, and

$$\frac{d\,e^{5x^2}}{dx} = 10xe^{5x^2}$$

Notice that the function which acts as an index - in this case $5x^2$ - is not changed but the entire function e^{5x^2} is multiplied by the derivative of the index.

Example 2 : If $f(x) = e^{4x}$, find $f'(x)$.
$f'(x) = 4e^{4x}$.

Example 3 : Find the derivative of the function $y = \log(6x+3)$. Let $p(x) = 6x+3$. Then $p'(x) = 6$ so that

$$\frac{d\,\log(6x+3)}{dx} = \frac{6}{6x+3}$$

Example 4 : Find the derivative of $f(x) = e^{3x}$.

$$f'(x) = 3e^{3x}$$

Example 5 : Find the derivative of $g(x) = \log(2x^3 + 3)$.

$$g'(x) = \frac{6x^2}{2x^3 + 3}$$

Exercises:

1. Differentiate the following with respect to x.

   (a) e^{5x}
   (b) e^{-2x}
   (c) e^{4x^2}
   (d) e^{6-x}
   (e) e^{4-2x^3}
   (f) $\log(4x + 1)$
   (g) $\log(3x^2 - 2)$
   (h) $\log x^3$
   (i) $\log 3x - 4$
   (j) $\log e^{4x}$

Section 2 Trigonometric Functions

The other special functions that you need to know how to differentiate are the trig functions. The rules are:

$$\begin{aligned} \frac{d \sin x}{dx} &= \cos x \\ \frac{d \cos x}{dx} &= -\sin x \\ \frac{d \tan x}{dx} &= \sec^2 x \end{aligned}$$

These can be generalized to

$$\begin{aligned} \frac{d \sin k(x)}{dx} &= k'(x) \cos k(x) \\ \frac{d \cos l(x)}{dx} &= -l'(x) \sin l(x) \\ \frac{d \tan m(x)}{dx} &= m'(x) \sec^2 m(x) \end{aligned}$$

We will show you how to derive these in the next worksheet. The main ones to remember are those for $\sin x, \cos x$, and $\tan x$.

Example 1 : Find the derivative of $f(x) = \sin 3x$. Then $f'(x) = 3\cos 3x$.

Example 2 : If $g(x) = \cos(x^2)$, then $g'(x) = -2x\sin(x^2)$.

Example 3 : If $h(x) = \tan(3x+2)$, then $h'(x) = 3\sec^2(3x+2)$.

Note: The functions $\sin^2 x$ and $\sin x^2$ are different functions. The notation $\sin^2 x$ means $(\sin x)^2$, and $\sin x^2$ means $\sin(x^2)$. They are both composite functions but they are not equal. Note also that $\sin^2 x$ does not mean $\sin(\sin x)$. This notation holds for the other trig functions as well.

Exercises:

1. Differentiate the following with respect to x.

   (a) $\sin x$

   (b) $\sin 4x$

   (c) $\sin 3x^2$

   (d) $\cos 5x$

   (e) $\cos x^2$

   (f) $\tan x$

   (g) $\tan 4x$

   (h) $\tan(6x+1)$

   (i) $\cos(-3x)$

   (j) $\sin(-6x)$

Exercises for Worksheet 3.10

1. Differentiate each of the following functions:

   (a) $f(x) = \log|x|$

   (b) $f(x) = \log(2x - 3)$

   (c) $f(x) = \log|x^2 - 3x + 2|$

   (d) $f(x) = 2e^{x^2}$

   (e) $f(x) = -\frac{1}{3}e^{x^2+2x-1}$

   (f) $f(x) = \sin 3x$

   (g) $f(x) = 3\cos\frac{x}{2}$

   (h) $f(x) = \frac{1}{2}\tan x^2$

   (i) $f(x) = \log x^3 - 2\sin 3x$

   (j) $f(x) = -\frac{1}{2}\cos 2x + \log|3x^2 - 1|$

2. (a) Absolute-value signs often enclose logarithmic expressions. But why? Using your calculator, complete the table below:

x	3.0	2.0	1.0	0.1	0.0	-0.1	-1.0	-2.0
$\ln x$								
$\ln\|x\|$								

   (b) Plot $\ln x$ and $\ln|x|$.

   (c) Why might $y = \ln|x|$ be a better representation of $\int \frac{1}{x}\,dx$ than $y = \ln x$?

3. (a) i. Find the slope of the function $f(x) = 1 - e^x$ at the point where it crosses the x-axis.

   ii. Find the equation of the tangent line to the curve at this point.

   iii. Find the equation of the normal at this point.

   (b) The world's population in 1975 was estimated at 4.1 billion. If the rate of population growth is 0.02, then the population $P(t)$ in billions is given by $P = 4.1e^{0.02t}$, where t is in years.

   i. Calculate the number of years it will take for the population to double.

   ii. Find $\frac{dP}{dt}$, $\left.\frac{dP}{dt}\right]_{t=0}$, and $\left.\frac{dP}{dt}\right]_{t=15}$. What do these quantities represent?

Answers to Test Three
and
Exercises from Worksheets 3.1 - 3.10

Answers to Test Three

1. (a) $9x^2 + 2$
   (b) $(2x+1)^2$
2. (a) $[-4, 0] \cup (2, 4]$
   (b) $x = -4$
3. (a) $\frac{\pi}{3}$
   (b) $\phi = \frac{\pi}{6}$
4. (a) $-\frac{1}{\sqrt{2}}$
   (b) $\sqrt{3}$
5. (a) $k = 2$
   (b) $u = -4$, $y = -6$
6. (a) 47
   (b) 10
7. (a) 10
   (b) No
8. (a) $3x^2 + 6x$
   (b) $(0, 0)$, $(-2, 4)$
9. (a) Max
   (b) $x = -\frac{1}{3}$
10. (a) $5\cos(5x+2)$
    (b) $3e^x$

Worksheet 3.1

---

1. (a) Output $=$ $50 + 2\times$ Input
   (b) Output $=$ Input $\times$ Input
   (c) $x \doteq 35.74$
   (d) 4.75
   (e) $37\frac{7}{9}$
   (f) 13
   (g) $-4\frac{2}{3}$
   (h) $(x+h)^2 - 2(x+h) + 3$
   (i) $6x - 2 + 3h$
   (j) 10.003
2. (a) i. $\frac{1}{x^2}$
      ii. $\frac{1}{x^2}$
   (b) i. $3(x-3)^2$
      ii. $3x^2 - 3$
   (c) $(2x+1)^2$
   (d) i. $f(2) = 4$
      ii. $f(x+h) = (x+h)^2$
      iii. $f(2x) = 4x^2$
      iv. $f(x+1) = (x+1)^2$
   (e) i. $f(\frac{1}{2}) = \frac{1}{2}$
      ii. $f(3+x) = \frac{1}{2x+7}$
      iii. $f(x^2) = \frac{1}{2x^2+1}$

3. (a) i. If the input is x then the output is $(3x+4)^2$
   ii. 4

   (b) i. If the input is x, then the output is $\sqrt{x-2}+5$
   ii. 5

   (c) $C = 27M$

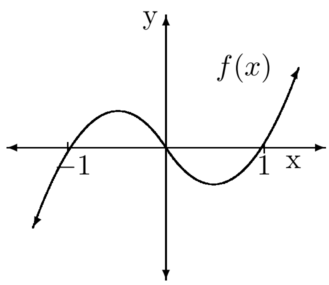
Worksheet 3.2

---

1. (a) $(-3, \infty)$
   (b) $(-\infty, -2)$ and $[1, 2)$
   (c) y-intercept -3; x-intercept $\frac{3}{2}$.
   (d) y-intercept -4; x-intercepts 4 and -1.
   (e)

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -(x^3 - x) \\ &= -f(x) \end{aligned}$$

   so $f(x)$ is odd.



2. (a) Circle
   (b) Parabola
   (c) Hyperbola
   (d) Circle
   (e) Striaght line
   (f) i. even
   ii. odd
   iii. neither

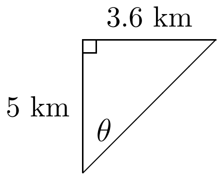
Worksheet 3.3

1. (a) i. $150°$ ii. $240°$ iii. $\frac{4\pi}{9}$ iv. $\frac{7\pi}{30} = 0.733$

   (b) i. $\frac{1}{\sqrt{2}}$ ii. $\frac{\sqrt{3}}{2}$ iii. $\sqrt{3}$

2. (a) $\sqrt{108}$ (b) $\frac{\pi}{3}$

3. (a)



   (b) $35.75°$

Worksheet 3.4

1. (a) $\frac{1}{\sqrt{2}}$
   (b) $\frac{1}{\sqrt{3}}$
   (c) $\frac{1}{\sqrt{2}}$
   (d) $-\frac{1}{2}$
   (e) $\frac{1}{2}$
   (f) $-\sqrt{3}$

2. (a) $\frac{\sqrt{3}+1}{2\sqrt{2}}$
   (b) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

3. (a) $\frac{\pi}{3}, \frac{4\pi}{3}$
   (b) $\frac{\pi}{4}, \frac{3\pi}{4}$
   (c) $\frac{3\pi}{4}, \frac{5\pi}{4}$

Worksheet 3.5

1. (a) None
   (b) One
   (c) Infinite
   (d) One
   (e) None

2. (a) $x = 8$, $y = 13$
   (b) $x = \frac{11}{5}, y = \frac{2}{5}$
   (c) $x = \frac{1}{17}$, $y = -\frac{5}{17}$

3. Peter is 15, Anneka is 9

Worksheet 3.6

1. (a) Arithmetic (b) Geometric (c) Neither (d) Neither (e) Arithmetic

2. (a) $T_6 = 15$, $T_{20} = 99$, $S_{10} = 120$
   (b) $T_6 = \log 7 + 5\log 2$, $T_{20} = \log 7 + 19\log 2$, $S_{10} = \frac{10}{2}(2\log 7 + 9\log 2)$
   (c) $T_6 = 2$, $T_{20} = 2^{15}$, $S_{10} = \frac{1}{16}(2^{10} - 1)$
   (d) $T_6 = (0.5)(0.9)^5$, $T_{20} = (0.5)(0.9)^{19}$, $S_{10} = 5(1 - .9^{10})$
   (e) $T_6 = -2$, $T_{20} = -\frac{1}{2^{13}}$, $S_{10} = \frac{128}{3}(1 + 2^{-10})$

3. (a) $a = 506$, $d = -18$ (b) $81(1 - (\frac{1}{3})^5)$ (c) $T_2 = (\frac{9}{4})^3$ (d) $n^2 + 6n$

4. (a) \$437,988.84 (b) 5 metres

Worksheet 3.7

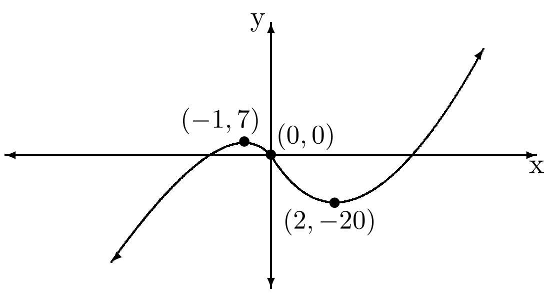
1. (a) 10
   (b) 6
   (c) $\frac{1}{3}$
   (d) $\frac{25}{29}$
   (e) 0

2. (a) Continuous
   (b) Continuous
   (c) Continuous
   (d) Not continuous
   (e) Not continuous

Worksheet 3.8

2. (a) $2x + 6$
   (b) $21x^2 - 10x + 9$
   (c) $\frac{1}{2\sqrt{x}} + 8$
   (d) $-6x^{-3} - x^{-2}$
   (e) $-\frac{2}{x^3} - \frac{1}{x^2} + 6$

3. (a)



(b) $T'(s) = -3$. The temperature is dropping 3 degrees for every km above sea level.

Worksheet 3.9

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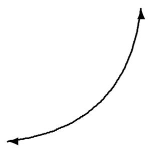
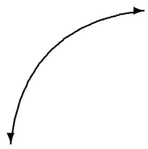
1. (a) $\Delta > 0$, 2 roots
   (b) $\Delta < 0$, no roots
   (c) $\Delta > 0$, 2 roots
   (d) $\Delta = 0$, 1 root
   (e) $\Delta < 0$, no roots

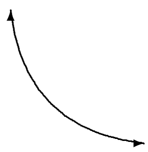
2. (a) Concave up and increasing.



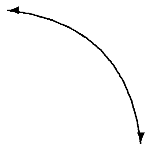
   (b) Concave down and increasing.



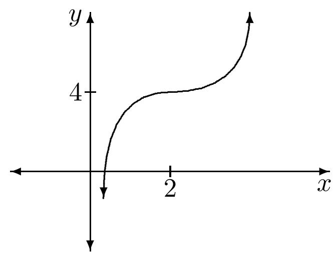
   (c) Concave up and decreasing.



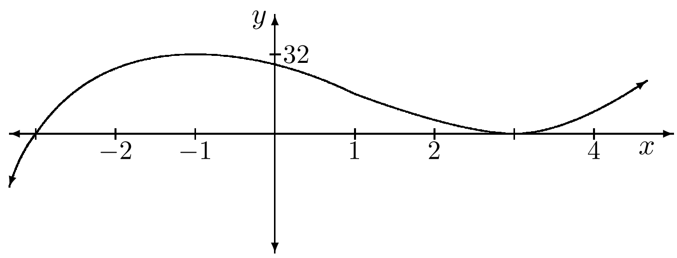
   (d) Concave down and decreasing.



   (e) Change of concavity at $(2, 4)$.



3. (a) $f(x) = (x-3)^2(x+3)$. Intercepts at $x = 3, -3$. $f'(x) = 3(x-3)(x+1)$. Stationary points at $x = 3, -1$. $f''(x) = 6x - 6$. $f''(3) > 0$ so there is a minimum point at $(3, 0)$. $f''(-1) < 0$ so there is a maximum point at $(-1, 32)$.



   (b) $\frac{dD}{dt} > 0$ and $\frac{d^2D}{dt^2} < 0$

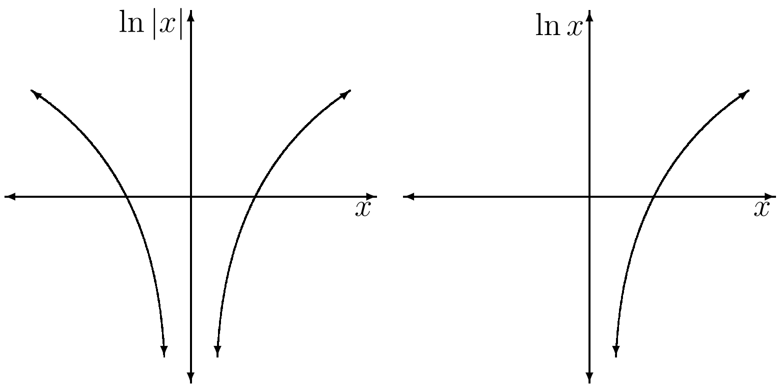
Worksheet 3.10

1. (a) $\frac{1}{x}$
   (b) $\frac{2}{2x-3}$
   (c) $\frac{2x-3}{x^2-3x+2}$
   (d) $4xe^{x^2}$
   (e) $-\frac{1}{3}(2x+2)e^{x^2+2x-1}$
   (f) $3\cos 3x$
   (g) $-\frac{3}{2}\sin\frac{x}{2}$
   (h) $x\sec^2 x^2$
   (i) $\frac{3}{x} - 6\cos 3x$
   (j) $\sin 2x + \frac{6x}{3x^2-1}$

2. (a)

x	3.0	2.0	1.0	0.1	0.0	-0.1	-1.0	-2.0
$\ln x$	1.098	0.69	0	-2.3	-	-	-	-
$\ln\|x\|$	1.098	0.69	0	-2.3	-	-2.3	0	0.69

   (b)



   (c) $g(x) = \ln|x|$ is defined on the interval $(-\infty, 0)$ and $(0,\infty)$, while $h(x) = \ln x$ is defined on $(0,\infty)$ only. Since $f(x) = \frac{1}{x}$ is defined on the same domain as $g(x)$, then $g(x)$ is a better representation of the integral than $h(x)$.

3. (a) i. -1
      ii. $y = -x$
      iii. $y = x$
   (b) i. $\frac{\ln 2}{0.02} \approx 35$
      ii. $\frac{dP}{dt} = 0.082e^{0.02t}$. This represents the rate of change of population per year (in billions of people per year).
      $\frac{dP}{dt}\Big]_{t=0} = 0.082$
      In 1975 the population was increasing at a rate of 82 million people per year.
      $\frac{dP}{dt}\Big]_{t=15} = 0.1107$
      In 1990 the population was increasing at a rate of 110.7 million people per year.