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Test Two

This is a self-diagnostic test. Every pair of questions relates to a worksheet in a series available in the MUMS the WORD series. For example question 5 relates to worksheet 2.5 *Arithmetic with Surds*. If you score 100% on this test and test 1 then we feel you are adequately prepared for your introductory mathematics course. For those of you who had trouble with a few of the questions, we recommend working through the appproriate worksheets and associated computer aided learning packages in this series.

1. (a) Find the highest common factor of $6uvw$ and $18uv$, where u, v, and w are prime numbers.
   (b) Factorize $3xy + 6y + 3y^2$
2. (a) Solve for x, $\frac{x}{4} = 3$
   (b) Solve for y, $6(y + 2) = 30$
3. (a) Simplify $\frac{2}{x+2} + \frac{3}{x}$
   (b) Solve for x in the equation $\frac{2x+2}{3x+5} = 2$
4. (a) Draw $x > -3$ on the number line.
   (b) Rewrite $|x| > 3$ without the absolute value signs.
5. (a) Simplify $(1 + \sqrt{2})(1 - \sqrt{2})$
   (b) Rationalize the denominator: $\frac{1}{2+\sqrt{2}}$
6. (a) Factorize $x^2 + 6x + 8$
   (b) Simplify $\frac{x+3}{x^2-9}$
7. (a) Is $x - 3$ a factor of $x^3 - 27$?
   (b) Factorize $x^3 + x^2 + x - 14$ as far as you can with integer coefficients.
8. (a) If $\log_2 \frac{x}{8} = 3$, what is x?
   (b) Simplify $3 \log e^y$
9. (a) In diagram A below, what is θ?
   (b) In diagram B below, what is $\sin\theta$, $\cos\theta$, and $\tan\theta$? What is θ?

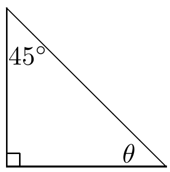


Diagram A

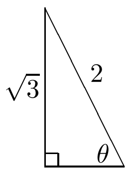


Diagram B

10. (a) If I drew $y = 5x + 3$ on graph paper, what would it look like?
    (b) What is the slope and the y-intercept of the line $3y = 5x + 2$?

Worksheet 2.1 Factors of Algebraic Expressions

Section 1 Finding Common Terms

In worksheet 1.2 we talked about factors of whole numbers. Remember, if $a \times b = ab$ then a is a factor of ab and b is a factor of ab. In a similar way we can look at the factors of an algebraic expression. So, for instance, $3uv$ has factors $1, 3, u, v$ and combinations of these like $3u, 3v, uv$ and of course $3uv$.

Other examples:-

1. $2ab = 2 \times a \times b$ has factors $1, 2, a, b, 2a, 2b, ab$ and $2ab$.
2. uvz has factors $u, v, z, uv, uz, vz, 1, uvz$.
3. xy has factors $1, x, y$ and xy.

In simplifying algebraic expressions we need to find common factors. Given two terms in an algebraic expression the common factors are those things which divide both of them.

Example 1 :

(a) $3uv$ and $6u$ have common factors $1, 3, u$ and $3u$.

(b) $2xy$ and $4xyz$ have common factors $1, 2, x, y, 2x, 2y, 2xy$ and xy.

The highest common factor is, as was the case with numbers, the biggest or largest factor that divides two expressions. So the highest common factor of $3uv$ and $6u$ (from example 1(a)) is $3u$; the highest common factor of $2xy$ and $4xyz$ (from example 1(b)) is $2xy$.

As with whole numbers we can also find the smallest algebraic expression that is a multiple of two expressions. This is called the lowest common multiple.

Example 2 :

$3uv$ and $6u$ have as their lowest common multiple $6uv$ since both $3uv$ and $6u$ divide into $6uv$ and do not both divide into a smaller expression.

To find the lowest common multiple we take the highest common factor and then multiply it by whatever is missing from each expression. From the example above $3uv$ and $6u$ have highest common factor $3u$. Now $3uv = 3u \times v$ and $6u = 3u \times 2$ so the lowest common multiple is $3u \times v \times 2 = 6uv$.

Example 3 : $9xy$ and $15xz$ have highest common factor $3x$.
$9xy = 3x \times 3y$ and $15xz = 3x \times 5z$ so the lowest common multiple is
$3x \times 3y \times 5z = 45xyz$.

Example 4 : $6a$ and $5b$ have highest common factor 1. So their lowest common multiple is $1 \times 6a \times 5b = 30ab$.

Exercises:

1. Find the highest common factor for each of the following:
   (a) $6x$, $18y$
   (b) $12mn$, $8m$
   (c) $3uv$, $4uw$
   (d) $18mp$, $9mn$
   (e) $27xyz$, $45xz$
2. Find the lowest common multiple for each of the following:
   (a) $6x$, $4xy$
   (b) $12xy$, $8xy$
   (c) $16mn$, $12np$
   (d) $24xyz$, $16yz$
   (e) $3m$, $45n$

Section 2 Simple Factoring

Sometimes in simplifying algebraic expressions or equations we would like to factorize them. This is a process which turns a sum into a product by removing common factors and placing them outside brackets. You have already seen an example of this.

When working out the perimeter of a paddock in the last worksheet we wanted two times the length plus two times the width. We wrote this as

$$P = 2l + 2w$$

By factorizing we can make a slightly tidier sum:

$$P = 2(l + w)$$

As 2 is a common factor to both terms it is placed outside of the brackets and the rest is left as a sum.

When factorizing algebraic expressions we look for the common factors in the terms and take these outside of the brackets to form a product as in the above example.

Example 1 :

$$9x + 24y = 3(3x + 8y)$$

Example 2 :

$$9x^2 + 3x + 15x^3 = 3(3x^2 + x + 5x^3)$$

But the terms inside the brackets still have x as a common factor:

$$9x^2 + 3x + 15x^3 = 3x(3x + 1 + 5x^2)$$

This is where we would stop since the terms inside the brackets have no further common factors.

Example 3 :

$$2ab^2 + ab^2c + 3ab = ab(2b + bc + 3)$$

Example 4 :

$$-2xy^2 - 4x^2y = -2xy(y + 2x)$$

Note that, by the laws mentioned in the last worksheet, the negative sign in front of the brackets will carry through the brackets, changing the sign of everything inside the brackets. Sometimes the common factor is not a simple multiple of numbers and letters but may in itself be a sum.

Example 5 : Simplify $5(x + 2) + y(x + 2) = (5 + y)(x + 2)$. We note that $(x + 2)$ is a common factor, so we put $(x + 2)$ out the front:

$$5(x + 2) + y(x + 2) = (x + 2)(5 + y)$$

Example 6 : $7(y + 1) - x(y + 1) = (y + 1)(7 - x)$

Exercises:

1. Factorize the following expressions:

   (a) $7x + 4$
   (b) $20x - 10$
   (c) $18xy - 3yz$
   (d) $12mn + 18mp$
   (e) $16m^2 - 4m$
   (f) $3x^2 + 6x - 18$
   (g) $-6x - 24$
   (h) $-2xy - 8x$
   (i) $24mn - 16m^2n$
   (j) $-x^2y - y^2x$
   (k) $12m^2n + 24m^2n^2$
   (l) $72y^2p - 18y^3p^2$

2. Factorize the following expressions:

   (a) $4(x + 3) + m(x + 3)$
   (b) $x(x - 1) + 5(x - 1)$
   (c) $y(y + 4) - 6(y + 4)$
   (d) $x^2(x + 7) + x(x + 7)$
   (e) $3x(x - 4) - 7(x - 4)$

Section 3 Algebraic Fractions

One use of the factorization of algebraic expressions or of being able to find common algebraic factors is to simplify algebraic fractions. Using the same method as with ordinary fractions we can cancel out common factors in algebraic fractions to make a simpler equivalent fraction.

Example 1 :

$$\frac{x}{2x} = \frac{1\times \not{x}}{2\times \not{x}} = \frac{1}{2}$$

by cancelling out the common x in the numerator and denominator.

Example 2 :

$$\frac{5x^2y}{15xy} = \frac{5xy \times x}{5xy \times 3} = \frac{x}{3}$$

noting that $5xy$ is a common factor.

Example 3 :

$$\frac{4a+2ab}{2a} = \frac{2a(2+b)}{2a} = 2+b$$

noting that $2a$ is a factor in common for the two terms in the sum and then cancelling.

Example 4 :

$$\begin{aligned} \frac{7x^2}{5y} \times \frac{15yz}{x} &= \frac{x \times 7x}{5y} \times \frac{5y \times 3z}{x} \\ &= \frac{\not{x} \times 7x \times \not{5}y \times 3z}{\not{5}y \times \not{x}} \\ &= 7x \times 3z \\ &= 21xz \end{aligned}$$

Note that, when either the numerator or denominator are completely cancelled, they become 1, not 0.

Example 5 :

$$\begin{aligned} \frac{x}{3} \div \frac{2x^2}{3} &= \frac{x}{3} \times \frac{3}{2x^2} \\ &= \frac{1}{2x} \end{aligned}$$

Example 6 :

$$\begin{aligned} \frac{6x+18}{20} \div \frac{3x+9}{15} &= \frac{6(x+3)}{20} \times \frac{15}{3(x+3)} \\ &= \frac{6 \times 15}{20 \times 3} \\ &= \frac{3 \times \not{2} \times \not{5} \times \not{3}}{\not{5} \times 2 \times \not{2} \times \not{3}} \\ &= \frac{3}{2} \\ &= 1\frac{1}{2} \end{aligned}$$

Exercises:

1. Simplify the following:

    (a) $\frac{3x}{15}$

    (b) $\frac{2x+10}{4}$

    (c) $\frac{5x+20}{x+4}$

    (d) $\frac{x^2-4x}{x-4}$

    (e) $\frac{3x^2-9x}{2x-6}$

    (f) $\frac{9x+27}{9x+18}$

    (g) $\frac{6ab+2a}{2b}$

    (h) $\frac{16m^2n-8mn}{12m-6}$

    (i) $\frac{4mnp-8mp}{12mn}$

2. Simplify the following:

    (a) $\frac{3x+9}{14} \times \frac{7x+21}{x+3}$

    (b) $\frac{x^2-5x}{2x+10} \times \frac{3x+15}{4x}$

    (c) $\frac{3mp+4p}{8p} \times \frac{12p^2}{3m+4}$

    (d) $\frac{16}{2mp+4m} \times \frac{6m^2+8m}{12}$

    (e) $\frac{24x-8}{12} \div \frac{9x-3}{6}$

    (f) $\frac{x^2+2x}{5} \div \frac{2x+4}{20}$

    (g) $\frac{p^2+pq}{7p} \div \frac{8p+8q}{21q}$

    (h) $\frac{5xy-15y}{4x-12} \div \frac{6y^2}{x+y}$

Exercises for Worksheet 2.1

1. What are the factors of 18?
2. What are the common factors of:
   (a) 16 and 24
   (b) $6tm$ and $14t^2$
3. What is the highest common factor of
   (a) 12 and 32
   (b) 24 and 40
   (c) 5 and 13
4. Factorize the following algebraic expressions
   (a) $-3x + 21$
   (b) $6x^2 + 3x$
   (c) $18x^2 + 12xy$
   (d) $6tm - 24m^2$
   (e) $8x + 12y + 10x + 15y$
   (f) $x^2 - 7x + 3x - 21$
5. Simplify the following
   (a) $\frac{3x+12}{3}$
   (b) $\frac{6xy+18x}{12}$
   (c) $\frac{7mn}{24} \times \frac{8}{m}$
   (d) $\frac{4x+20}{5} \div \frac{8x+40}{20}$
   (e) $\frac{6xy^2}{7} \times \frac{21x^2}{y} \div \frac{32xy^2}{91}$
   (f) $\frac{12p^2q^2}{5} \times \frac{15}{4pq} \div 3$

Worksheet 2.2 Solving Equations in One Variable

Section 1 Simple Examples

You are on your way to Brisbane from Sydney, and you know that the trip is 1100 km. You pass a sign that says that Brisbane is now 200 km away. How far have you travelled? We can see the answer quickly, but we shall write the information down in the form of a mathematical expression. Let x be the distance travelled so far. Then

$$x + 200 = 1100$$

This is the information that we are given. To find the answer we are to solve an equation in one variable, x.

$$\begin{aligned} x + 200 &= 1100 \\ x + 200 - 200 &= 1100 - 200 \qquad \text{subtract 200 from both sides} \\ x &= 900 \end{aligned}$$

The middle step emphasizes what we do in the calculation. We wanted x by itself on one side of the equation, so we subtracted 200 from each side of the equation. An equation is like a balancing beam in the sense that if you do anything to one side of the equation, you must also do it to the other.

Example 1 : If $x + 2 = 3$, what is x? Clearly, $x = 1$, but again we follow through the calculation.

$$\begin{aligned} x + 2 &= 3 \\ x + \not{2} - \not{2} &= 3 - 2 \\ x &= 1 \end{aligned}$$

Example 2 : If $x - 5 = -12$, what is x?

$$\begin{aligned} x - \not{5} + \not{5} &= -12 + 5 \qquad \text{adding 5 to both sides} \\ x &= -7 \end{aligned}$$

Example 3 : Solve for x in $-8 = -10 + x$.

$$\begin{aligned} -8 &= -10 + x \\ -8 + 10 &= -10 + x + 10 \\ x &= 2 \end{aligned}$$

In each of the above calculations, we worked on the equation by trying to get x on one side and all the rest of the expression on the other side. This is a basic method for solving equations in one variable. To simplify the expression involving x we simply undid whatever was done to x by applying the opposite arithmetic operation to leave us with just an x on one side. *Most importantly*, we need to remember that whatever we do to one side of the equation, we must do to the other. For example, if we subtract 5 from one side of the equation, we must subtract 5 from the other.

Exercises:

1. Solve the following equations:

   (a) $x + 9 = 20$

   (b) $x - 8 = 10$

   (c) $x + 1.6 = 2.4$

   (d) $x - \frac{3}{4} = 1\frac{1}{2}$

   (e) $y - 2 = -8$

   (f) $m + 6 = -4$

   (g) $m - 7.1 = -8.4$

   (h) $t - 2.4 = -1$

   (i) $x + 1\frac{1}{2} = 2$

   (j) $y + 80 = 120$

Section 2 Equations Involving Multiplication

In a similar way to the above we can deal with solving equations such as

$$5x = 2$$

The arithmetic operation is now multiplication, whose inverse is division. The inverse of multiplying by 5 is dividing by 5.

$$\begin{aligned} 5x &= 2 \quad \text{becomes} \\ \frac{\not{5}}{\not{5}}x &= \frac{2}{5} \quad \text{dividing both sides by 5} \\ 1x &= \frac{2}{5} \\ x &= \frac{2}{5} \end{aligned}$$

Example 1 :

$$
\begin{aligned}
\frac{x}{3} &= 2 \quad \text{becomes} \\
\frac{x}{\cancel{3}} \times \cancel{3} &= 2 \times 3 \quad \text{multiplying both sides by 3} \\
x &= 6
\end{aligned}
$$

Example 2 :

$$
\begin{aligned}
\frac{2}{7}x &= 4 \\
\cancel{7} \times \frac{2}{\cancel{7}}x &= 4 \times 7 \qquad \text{(multiplying both sides by 7)} \\
2x &= 28 \\
\frac{\cancel{2}x}{\cancel{2}} &= \frac{28}{2} \qquad \text{(dividing both sides by 2)} \\
x &= 14
\end{aligned}
$$

Exercises:

1. Solve the following equations:

(a) $4x = 20$

(b) $6x = 24$

(c) $\frac{x}{3} = 5$

(d) $\frac{x}{5} = 1.2$

(e) $2x = 11$

(f) $3x = -24$

(g) $\frac{x}{2} = -7$

(h) $\frac{3x}{4} = -12$

(i) $\frac{2}{5}m = 10$

(j) $\frac{3m}{4} = -6$

Section 3 Multiple Terms

The puzzle page in the newspapers sometimes has puzzles like this example:

Example 1 : Jean is 7 years older than half of Tom's age. If Jean is 35, how old is Tom? We now write the information that we are given in the form of an algebraic expression. Let Jean's age be J and Tom's T. Then

$$J = \frac{1}{2}T + 7$$

Putting in J as 35, we get

$$35 = \frac{1}{2}T + 7$$

What is T? Notice that we now have a combination of arithmetic operations to deal with. It doesn't matter what order you do them in so long as you remember to include the whole of each side of the equation in the undoing, and whatever you do to one side you must do the same to the other side. We will solve for T in two ways.

Solution 1

$$\begin{aligned}
35 &= \frac{1}{2}T + 7 \\
35 - 7 &= \frac{1}{2}T + 7 - 7 \quad \text{(subtract 7 from both sides)} \\
28 &= \frac{1}{2}T \\
2 \times 28 &= 2 \times \frac{1}{2}T \quad \text{(multiply both sides by 2)} \\
56 &= T
\end{aligned}$$

Solution 2

$$\begin{aligned}
35 &= \frac{1}{2}T + 7 \\
2 \times 35 &= 2 \times (\frac{1}{2}T + 7) \quad \text{(multiply both \underline{sides} by 2)} \\
70 &= T + 2 \times 7 \quad \text{(multiply out the brackets)} \\
70 &= T + 14 \\
70 - 14 &= T + 14 - 14 \\
56 &= T
\end{aligned}$$

Note: When multiplying through by a number it is important to multiply every term on both sides of the equation. On the whole, it is probably easier to undo addition and subtraction first.

Example 2 :

$$\begin{aligned}
3x - 5 &= 16 \\
3x - 5 + 5 &= 16 + 5 \qquad \text{(add 5 to both sides)} \\
3x &= 21 \\
\frac{3x}{3} &= \frac{21}{3} \\
x &= 7
\end{aligned}$$

Occasionally you may be asked to solve an equation in one variable where the variable appears on both sides of the equation.

Example 3 : Solve

$$3x + 5 = 2x + 1 \qquad \text{for } x$$

We will undo the operations in a way that puts all the x parts of the expression on one side of the equation.

$$\begin{aligned}
3x + 5 &= 2x + 1 \\
3x + 5 - 2x &= 2x + 1 - 2x \qquad \text{(subtract } 2x \text{ from both sides)} \\
x + 5 &= 1 \\
x + 5 - 5 &= 1 - 5 \qquad \text{(subtract 5 from both sides)} \\
x &= -4
\end{aligned}$$

Sometimes the expressions will be more complex and could involve brackets. In these cases we expand out the brackets and proceed.

Example 4 :

$$\begin{aligned}
3(5 - x) - 2(5 + x) &= 3(x + 1) \\
15 - 3x - 10 - 2x &= 3x + 3 \qquad \text{(collect like terms)} \\
5 - 5x &= 3x + 3 \\
5 - 5x + 5x &= 3x + 3 + 5x \qquad \text{(add } 5x \text{ to both sides)} \\
5 &= 8x + 3 \\
5 - 3 &= 8x + 3 - 3 \qquad \text{(subtract 3 from both sides)} \\
2 &= 8x \\
\frac{2}{8} &= \frac{8x}{8} \qquad \text{(divide both sides by 8)} \\
\frac{1}{4} &= x
\end{aligned}$$

Once you feel confident in these processes there is no need to put in the intermediate steps illustrated in these examples.

Exercises:

1. Solve the following equations:

   (a) $2x - 1 = 9$

   (b) $\frac{y}{3} + 4 = 12$

   (c) $2(x + 1) - 7 = 5$

   (d) $4(y + 3) - 2y = 7$

   (e) $5(y + 2) - 4(y - 1) = 6$

   (f) $5(2 - x) - 3(4 - 2x) = 20$

   (g) $2m + 4 - 3m = 8(m - 1)$

   (h) $3m + 12 = 2(m - 3) + 4$

   (i) $\frac{x+1}{4} = 5$

   (j) $\frac{x}{5} + \frac{x}{3} = 10$

Exercises for Worksheet 2.2

1. Solve

   (a) $x + 4 = -7$
   (b) $2 - x = 13$
   (c) $15y = 45$
   (d) $-\frac{t}{2} = -9$
   (e) $3y - 20 = \frac{1}{2}$
   (f) $\frac{x+3}{2} = -1$
   (g) $3x + 2 = 4x - 7$
   (h) $\frac{x}{2} + 7 = \frac{3x}{4}$
   (i) $2x(x + 3) = 2x^2 + 15$
   (j) $(y + 7)(y + 7) = y^2$
   (k) $2x + 7 + 8x = 13$
   (l) $3(x + 1) + 4x = 26$
   (m) $8(m - 3) - 2(m - 2) = 20$
   (n) $\frac{y+3}{2} = \frac{y-4}{3}$
   (o) $3(4 - y) = 2(y + 5)$
   (p) $\frac{x}{7} = 3\frac{1}{7}$
   (q) $\frac{x+1}{2} = \frac{3}{4}$
   (r) $16t - 7 + 4t = 12t - 1$
   (s) $\frac{t}{4} + 3 = \frac{t}{8} - 1$
   (t) $8 = \frac{1}{3}T + 2$

2. (a) Three times a number is equal to the number decreased by two. What is the number?
   (b) The sum of two consecutive numbers is 93. What are the numbers?
   (c) The sum of two consecutive *even* numbers is 46. Find the numbers.
   (d) When the tax on cigarettes was increased by 5%, the price of a certain pack became \$5.60. What was the oringinal price?
   (e) In 1994, 15% of the women who went on maternity leave returned to full-time work, while in 1993 only 12% returned. If the number returning in each year was 6, how many left to have babies in 1994?
   (f) Two times a number is equal to six less than three times the number. What is the number?

Worksheet 2.3 Algebraic Fractions

Section 1 Factoring and Algebraic Fractions

As pointed out in worksheet 2.1, we can use factoring to simplify algebraic expressions, and in particular we can use it to simplify algebraic fractions. Calculations using algebraic functions are similar to calculations involving fractions. So when adding together fractions with different denominators, we must first find the lowest common multiple.

Example 1 :

$$\begin{aligned} \frac{a+b}{2} - \frac{2a}{5} &= \frac{5(a+b)}{5 \times 2} - \frac{2(2a)}{2 \times 5} \\ &= \frac{5a+5b}{10} - \frac{4a}{10} \\ &= \frac{5a+5b-4a}{10} \\ &= \frac{a+5b}{10} \end{aligned}$$

Example 2 :

$$\begin{aligned} \frac{3}{2x} - \frac{y+1}{3xy} &= \frac{3y \times 3}{3y \times 2x} - \frac{2(y+1)}{2 \times 3xy} \\ &= \frac{9y}{6xy} - \frac{2y+2}{6xy} \\ &= \frac{9y-(2y+2)}{6xy} \\ &= \frac{9y-2y-2}{6xy} \\ &= \frac{7y-2}{6xy} \end{aligned}$$

Example 3 :

$$\begin{aligned}
\frac{2}{y+1}+\frac{3}{y-1} &= \frac{2(y-1)}{(y+1)(y-1)}-\frac{3(y+1)}{(y-1)(y+1)} \\
&= \frac{2y-2}{(y+1)(y-1)}-\frac{3y+3}{(y-1)(y+1)} \\
&= \frac{2y-2-3y-3}{(y+1)(y-1)} \\
&= \frac{-y-5}{(y+1)(y-1)} \\
&= \frac{-(y+5)}{(y+1)(y-1)}
\end{aligned}$$

Sometimes it is difficult to find a simple expression that is a multiple of two algebraic expressions. When this is the case it is perfectly acceptable to multiply the two expressions together even though this will not necessarily form the smallest common multiple. You should check at the end of the calculation in the final fraction that there are no common factors in the numerator and denominator; if there are, you can always cancel them to give an equivalent but simpler fraction.

Exercises:

1. Simplify the following algebraic expressions:

   (a) $\frac{x}{3}+\frac{x}{2}$

   (b) $\frac{m}{7}-\frac{m}{5}$

   (c) $\frac{4t}{5}+\frac{t}{2}$

   (d) $\frac{m+1}{3}-\frac{m-2}{4}$

   (e) $\frac{3m+4}{7}+\frac{m-1}{2}$

   (f) $\frac{y}{y+1}-\frac{y}{y+3}$

   (g) $\frac{5}{t+1}+\frac{4}{t-3}$

   (h) $\frac{3m}{m+4}+\frac{4m}{m+5}$

   (i) $\frac{4}{y+1}-\frac{5}{y+2}$

   (j) $\frac{7}{4x}+\frac{2}{5xy}$

Section 2 Multiplication and Division

As in numerical fractions, the trick with simplifying the multiplication and division of algebraic fractions is to look for common factors both before and after calculation. Once common factors are cancelled out you get an equivalent fraction in its simplest form. Remember that dividing by a fraction is the same operation as multiplying by the reciprocal. That is

$$\frac{1}{\frac{1}{x}} = 1 \div \frac{1}{x} = 1 \times \frac{x}{1} = x$$

For example $\frac{1}{\frac{1}{6}}$ means how many 6ths are in one whole? The answer is 6.

Also, an algebraic expression in the numerator or denominator should be treated as if it were in brackets. For instance

$$-\frac{x+2}{4} = -\frac{(x+2)}{4} = \frac{-x-2}{4}$$

Example 1 :

$$\begin{aligned} \frac{x}{2} \div \frac{x}{8} &= \frac{x}{2} \times \frac{8}{x} \\ &= \frac{8x}{2x} \\ &= 4 \end{aligned}$$

Example 2 :

$$\begin{aligned} \frac{8}{x+2} \div \frac{7}{2x+4} &= \frac{8}{x+2} \times \frac{2x+4}{7} \\ &= \frac{8 \times 2(x+2)}{7(x+2)} \\ &= \frac{16}{7} \\ &= 2\frac{2}{7} \end{aligned}$$

Example 3 :

$$\begin{aligned} \frac{\frac{x+1}{4y}}{\frac{x+4}{8y}} &= \frac{x+1}{4y} \div \frac{x+4}{8y} \\ &= \frac{x+1}{4y} \times \frac{8y}{x+4} \\ &= \frac{8y(x+1)}{4y(x+4)} \\ &= \frac{2(x+1)}{x+4} \end{aligned}$$

Example 4 :

$$\frac{\frac{4}{xy}}{\frac{x+2}{6}} = \frac{4}{xy} \times \frac{6}{x+2} = \frac{24}{xy(x+2)}$$

Example 5 :

$$\frac{3}{xy} \times \frac{y}{6} = \frac{\not{3}}{x \times \not{y}} \times \frac{\not{y}}{\not{3} \times 2} = \frac{1}{2x}$$

Example 6 :

$$\frac{\left(\frac{x}{x-1} - \frac{y}{y-1}\right)}{\frac{3}{xy}} = \frac{\left(\frac{(y-1)x}{(y-1)(x-1)} - \frac{(x-1)y}{(x-1)(y-1)}\right)}{\frac{3}{xy}} = \frac{\left(\frac{xy-x-xy+y}{(x-1)(y-1)}\right)}{\frac{3}{xy}} = \frac{-x+y}{(x-1)(y-1)} \times \frac{xy}{3} = \frac{(y-x)xy}{3(x-1)(y-1)}$$

Exercises:

1. Simplify the following algebraic expressions:

   (a) $\frac{m}{16} \div \frac{5m}{12}$

   (b) $\frac{3m}{8} \div \frac{15m}{20}$

   (c) $\frac{6x+3}{8} \div \frac{2x+1}{12}$

   (d) $\frac{9xy}{7} \div \frac{6x}{3}$

   (e) $\frac{6pq}{5} \div \frac{12p}{7}$

   (f) $\frac{3(x+1)}{8} \div \frac{5(x+1)}{16}$

   (g) $\frac{\frac{4x}{7}}{\frac{6xy}{5}}$

   (h) $\frac{\frac{m+1}{2} - \frac{m-1}{3}}{\frac{m+1}{5}}$

   (i) $\frac{3}{pq} \times \frac{4p}{p+1}$

   (j) $\frac{8(x+3)}{9} \times \frac{12(x+1)}{4(x+3)}$

Section 3 Solving Equations

Sometimes we are asked to solve an equation for a particular variable. This means that only the variable should be on one side of an equality sign and the other information in the equation should be on the other side. This is similar to solving equations in one variable as in Worksheet 2.2. However, you may end up with an algebraic expression on one side involving other variables rather than just a number. You should attack these questions in the same way as solving equations for one variable.

The following examples and exercises use some of the techniques given in sections one and two of this worksheet.

Example 1 :

$$\frac{x-2}{3} + \frac{x+1}{5} = 3$$

Multiply each side by 15 - this will eliminate the fractions:

$$\begin{aligned} 15 \times (\frac{x-2}{3} + \frac{x+1}{5}) &= 15 \times 3 \\ 15\frac{x-2}{3} + 15\frac{x+1}{3} &= 45 \\ 5(x-2) + 3(x+1) &= 45 \\ 5x - 10 + 3x + 3 &= 45 \\ 8x - 7 &= 45 \\ 8x &= 52 \\ x &= \frac{52}{8} \\ &= \frac{13}{2} \end{aligned}$$

Example 2 :

Solve for x in terms of y.

$$\begin{aligned}
\frac{1+2xy}{y+1} &= 3x \\
(y+1) \times \frac{(1+2xy)}{(y+1)} &= (y+1)3x \quad \text{(multiplying both sides by } (y+1)) \\
1+2xy &= 3xy+3x \\
1 &= xy+3x \\
1 &= x(y+3) \qquad \text{(factoring to separate the } x) \\
x &= \frac{1}{y+3}
\end{aligned}$$

Example 3 :

Solve for y in terms of x.

$$\begin{aligned}
\frac{4+x}{3y^2} &= \frac{x+1}{2y} \\
\frac{6y^2(4+x)}{3y^2} &= \frac{6y^2(x+1)}{2y} \qquad \text{(multiply both sides by } 6y^2) \\
2(4+x) &= 3y(x+1) \\
\frac{2(4+x)}{(x+1)} &= 3y \qquad \text{isolate } y \\
y &= \frac{2(4+x)}{3(x+1)}
\end{aligned}$$

In the last two steps, we were aiming to make x the subject of the equation.

Exercises:

1. Solve for x:
   (a) $\frac{x+8}{5} - \frac{x-2}{3} = 4$
   (b) $\frac{x+1}{3} + \frac{x-4}{2} = 5$
   (c) $\frac{3(x-2)}{4} - \frac{2(x+1)}{5} = \frac{1}{10}$
   (d) $\frac{4}{x+1} + \frac{3}{x-4} = \frac{2}{x+1}$
   (e) $\frac{5}{x+3} + \frac{2}{2x+6} = 4$

2. Solve for x in terms of y:

   (a) $3xy = 8$
   (b) $\frac{4}{x+1} = \frac{3}{y+2}$
   (c) $4(y+1) - 3(x+5) = 8$
   (d) $\frac{1+y}{2+x} = 3y$
   (e) $5xy + 3xy^2 = 7$

Exercises for Worksheet 2.3

1. Simplify the following:

   (a) $\frac{x}{3} + \frac{x}{4}$

   (b) $\frac{2}{2xy} + \frac{4}{xy^3}$

   (c) $\frac{3x+1}{2} - (6x + 5)$

   (d) $\frac{3}{b-1} - \frac{4}{b-2}$

   (e) $\frac{\frac{2x+2}{y}}{\frac{x+1}{xy}}$

   (f) $\frac{2}{x^2-4x} + \frac{4}{x-4}$

   (g) $\frac{\frac{1}{x+1}}{2-1}$

   (h) $\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3}$

   (i) $\frac{4a}{7} + \frac{3a+5}{2} - \frac{3(a+2)}{4}$

   (j) $\frac{3p}{12} - (\frac{p}{2} - \frac{p}{4} + \frac{5p}{6})$

2. Simplify the follwing:

   (a) $\frac{4(x+1)}{3} - \frac{5(x-2)}{2}$

   (b) $\frac{x^2+3x}{x+4} \times \frac{2x+8}{5x}$

   (c) $\frac{8x-24}{4} \div \frac{x+7}{12}$

   (d) $\frac{y^2-6y}{y+5} \times \frac{3y+15}{2y-12}$

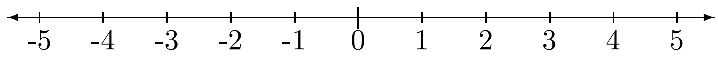
   (e) $\frac{5m-7}{4m+8} \div \frac{m+2}{3m+6}$

   (f) $\frac{6p-3}{4} \div \frac{-4p+2}{12}$

Worksheet 2.4 Introduction to Inequalities

Section 1 Inequalities

The sign $<$ stands for less than. It was introduced so that we could write in shorthand things like 3 is less than 5. This becomes $3 < 5$. The sign $>$ stands for greater than. In a similar way we can write $5 > 3$ in this shorthand form. These statements are called inequalities. Recall the number line (introduced in an earlier worksheet):



It is drawn so that the numbers increase from left to right. Alternatively the numbers decrease from right to left. Any number which lies to the right of another on the number line is greater than it and any number which lies to the left of another on the number line is less than it.

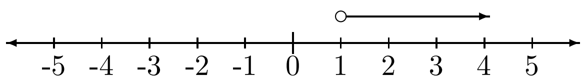
Example 1 :

- $-6 < 0$
- $5 > 2$
- $-5 < -4$
- $2.5 < 3$

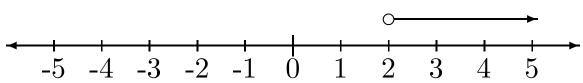
We can also use inequalities in algebraic expressions. So $21a$ is less than $30a$ is written

$$21a < 30a \quad \text{(only true if } a > 0\text{)}$$

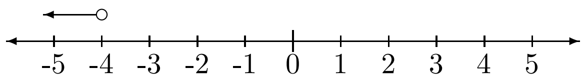
The expression $a > 1$ means that a is one of all the numbers to the right of 1 on the number line. We draw this using an open circle and an arrow heading to the right. The open circle sits over 1 and indicates that the actual number 1 is not included. That is a is bigger than 1 but not equal to 1:



Example 2 : $x > 2$



Example 3 : $x < -4$



This indicates all numbers to the left of -4 but not including -4.

If we wish to include -4 in the above example we would write $x \leq -4$. This is shorthand for less than or equal to -4.

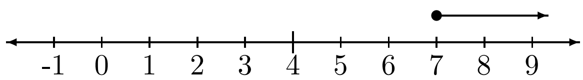
$$x \leq a \quad \Rightarrow \quad x < a \quad \text{or} \quad x = a$$

and

$$x \geq a \quad \Rightarrow \quad x > a \quad \text{or} \quad x = a$$

For $\Rightarrow$, read 'implies'. A closed circle or dot is used on the number line when the actual number is included in the inequality.

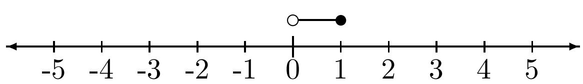
Example 4 : $x \geq 7$



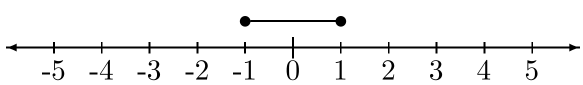
This indicates all the numbers to the right of 7 and 7 itself.

We can combine signs to make further algebraic expressions. For example $-3 \leq x < 2$ means all numbers greater than or equal to -3 but smaller than or equal to 2. That -3 and 2 and all the numbers in between. You may also see the notation $[-3, 2]$ used to represent this interval.

Example 5 : $0 < x \leq 1$ means all numbers greater than zero and less than or equal to one. In interval notation this is represented by $(0, 1]$. To draw this type of inequality on the number line we use a line drawn between the open or closed circles over the numbers. So $0 < x \leq 1$ is drawn as



Example 6 : $-1 \leq x \leq 1$



Example 7 : The number of matches in a box could be as little as 47 or as many as 58. If x stands for the number of matches in a box then we can write that

$$47 \leq x \leq 58$$

Exercises:

1. Write inequalities for the following:
   (a) Numbers less than or equal to 6
   (b) Numbers between -1 and 4 inclusive
   (c) Numbers between 2 and 5 exclusive
2. Which of the numbers indicated satisfy the accompanying inequality?
   (a) $x \geq 3$ $\quad -1, 2, 3, 3\frac{1}{2}, 4\frac{1}{4}$
   (b) $x < -4$ $\quad -2, -7, -8\frac{1}{2}, 0$
   (c) $-5 \leq x < 3$ $\quad -6, -4, 0, \frac{1}{2}, 3$
   (d) $2 < x \leq 5\frac{1}{2}$ $\quad 1\frac{1}{2}, 2\frac{1}{2}, 4, 5\frac{1}{4}, 5.5, 7$
3. Graph the following inequalities on number lines:
   (a) $x \leq 1$
   (b) $x > -4$
   (c) $-1 < x < 5$

Section 2 Solving Inequalities

Solving inequalities is similar to solving equations as in worksheet 2.2. We may add or subtract numbers or algebraic terms from both or all sides of the inequality to isolate the variable from the rest of the expression . Similarly we can multiply and divide each side with one very important qualification.

> When multiplying or dividing both sides of an inequality by a <u>negative number</u> the sign must be <u>reversed</u>.

Example 1 : Given $-2x < 4$ we want to solve for x. Divide both sides by -2 and reverse the inequality. We get

$$\frac{-2x}{-2} > \frac{4}{-2}$$

and after cancelling we get

$$x > -2$$

Only do this when multiplying or dividing both sides of the inequality by a negative number. Let's see what happens when we don't do this.

Example 2 : Observe that $-1 < 2$ is a true statement.

$$\begin{aligned} -1 &< 2 \\ \frac{-1}{-1} &< \frac{2}{-1} \\ 1 &< -2 \end{aligned}$$

We have ended up with $1 < -2$, which is a false statement.

Some examples of solving inequalities follow.

Example 3 :

$$\begin{aligned} x+3 &\geq 4 \\ x+3-3 &\geq 4-3 \\ x &\geq 1 \end{aligned}$$

Example 4 :

$$\begin{aligned} x-2 &\geq -5 \\ x-2+2 &\geq -5+2 \\ x &\geq -3 \end{aligned}$$

Example 5 :

$$\begin{aligned} 5x &\geq 10 \\ \frac{5}{5}x &\geq \frac{10}{5} \\ x &\geq 2 \end{aligned}$$

Example 6 :

$$\begin{aligned} 3-2x &\geq 5 \\ -3+3-2x &\geq -3+5 \\ -2x &\geq 2 \\ \frac{-2x}{-2} &\leq \frac{2}{-2} \\ x &\leq -1 \end{aligned}$$

Example 7 :

$$\begin{aligned} 5x+3 &\leq 2x+2 \\ 5x+3-2x &\leq 2x+2-2x \\ 3x+3 &\leq 2 \\ 3x &\leq 2-3 \\ 3x &\leq -1 \\ x &\leq \frac{-1}{3} \end{aligned}$$

Example 8 :

$$\begin{aligned} 3(x+\frac{2}{3}) &< 5(2x+5) \\ 3x+2 &< 10x+25 \\ 3x-10x+2 &< 25 \\ -7x &< 25-2 \\ -7x &< 23 \\ x &> \frac{23}{-7} \\ x &> -\frac{23}{7} \end{aligned}$$

Exercises:

1. Solve the following inequalities:

   (a) $x-7<5$

   (b) $2x+3>8$

   (c) $4(m+1)\leq m-3$

   (d) $\frac{m+1}{2}\geq 4$

   (e) $6-m<8-4m$

   (f) $3y+2\leq 5y+10$

(g) $\frac{y-1}{3} + 4 > \frac{y+1}{2}$

(h) $4(x + \frac{1}{2}) - 2(x + \frac{3}{2}) \leq 5$

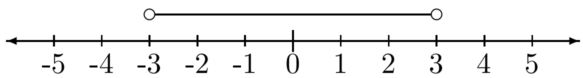
(i) $\frac{2x+1}{5} - \frac{x+3}{2} < 4x + 10$

(j) $4(m + 3) + 5(2m - 1) > 7m + 6$

Section 3 Absolute Values and Inequalities

The absolute value of a number was discussed in Worksheet 1.7. Recall that the absolute value of a, written $|a|$, is the distance in units that a is away from the origin. For example $|-7| = 7$. Alternatively you could define it as $|a| = +\sqrt{a^2}$. So when we combine absolute values and inequalities we are looking for all numbers that are either less than or greater than a certain distance away from the origin.

Example 1 : The equation $|x| < 3$ represents all the numbers whose distance away from the origin is less than 3 units. We could rewrite this inequality as $-3 < x < 3$. If we draw this on the number line we get

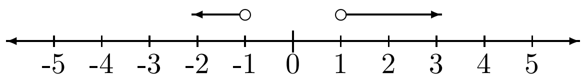


To solve inequalities involving absolute values we often need to rewrite the inequality without the absolute value signs as we have just done in the above example.

Example 2 : $|x| > 1$ can be rewritten as

$$x > 1 \quad \text{or} \quad x < -1$$

In other words all the numbers whose distance away from the origin is greater than 1 unit.



Notice that the solution is just the converse of $-1 \leq x \leq 1$. Since $|x| \leq 1$ is easier to solve than $|x| > 1$ we can begin by solving $-1 \leq x \leq 1$ and the solution will be its converse.

Example 3 :

$$\begin{aligned} |x| + 2 &\leq 4 \\ |x| + 2 - 2 &\leq 4 - 2 \\ |x| &\leq 2 \end{aligned}$$

this is the same as

$$-2 \leq x \leq 2$$

Example 4 :

$$|x + 3| < 5$$

$$-5 < x + 3 < 5$$

Here we note that the expression inside the absolute value sign is treated as a single entity, as if it were in brackets, and must be rewritten before you try to solve it. So

$$\begin{array}{rcl} -5 < & x + 3 & < 5 \\ -5 - 3 < & x + 3 - 3 & < 5 - 3 \\ -8 < & x & < 2 \end{array}$$

When dealing with a three-sided inequality like the above we follow the same rules as if it were two-sided, i.e. everything that we do to one side of the inequality we must do to all the other sides.

Example 5 : Solve:

$$|x - 2| \geq 4$$

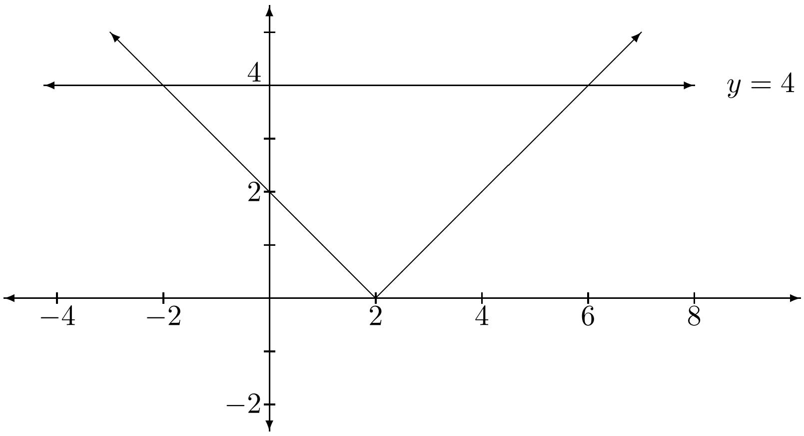
Consider the converse: $|x - 2| < 4$. Then

$$\begin{array}{rcl} -4 < & x - 2 & < 4 \\ -2 < & x & < 6 \end{array}$$

Hence the solution is the converse

$$x \leq -2 \quad \text{or} \quad x \geq 6$$

This can also be viewed graphically:



Notice that the function $y = |x - 2|$ is at or above the line $y = 4$ for $x \leq -2$ and $x \geq 6$. To graph $y = |x - 2|$, we set up a table of values as follows:

x	0	1	4	-4
y	2	1	2	6

Exercises:

1. Solve the following inequalities, and graph the solutions on a number line:

   (a) $|x| \leq 3$

   (b) $|x| > 4$

   (c) $|x| + 2 \leq 5$

   (d) $|x + 1| < 6$

   (e) $|x - 4| \geq 2$

Exercises for Worksheet 2.4

1. (a) Show the following inequalities on a number line:

    i. $x > 3$

    ii. $x \leq -2.5$

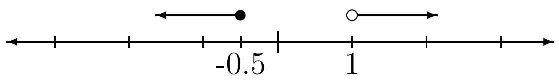
    iii. $-1 < x \leq 4$

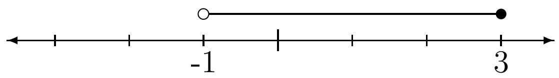
    iv. $x > 6$ or $x < -8$

    v. $[0, 5)$

    vi. $(-2, 4]$

   (b) Write down the appropriate inequalities from the following information.

    i. 

    ii. 

    iii. Sophie's blood alcohol level was at least 0.07

    iv. Tony had no more than 2 traffic offences.

    v. The number of drug offences was no more than 200.

    vi. To enter the theatre, you must be at least 18 years old or under 12 months old.

    vii. To join the veteran's team, you must be 35 or more.

   (c) i. Is $x = 1.5$ a solution to $7x + 5 > 19$?

    ii. Is $x = -2$ a solution to $4 - x < 6$?

2. Solve:

   (a) $x - 3 \leq 2$

   (b) $-y > 6$

   (c) $2x < 24$

   (d) $-1 < x + 2 < 5$

   (e) $2a + 4 > 3a - 11$

   (f) $-2x + 7 > 10$

   (g) $3(t + 5) \leq 2(t + 1)$

   (h) $\frac{1}{2}(x + 3) \geq \frac{3}{4}(x - 2)$

   (i) $|x| = 7$

   (j) $|6 - x| = 7$

   (k) $|y| > 2$

   (l) $|2x + 1| < 3$

   (m) $|x - 5| \geq 5$

   (n) $|x + 13| < -2$

Worksheet 2.5 Arithmetic with Surds

Section 1 Expanding Expressions Involving Surds

Fractional powers and the basic operations on them are introduced in worksheet 1.8. This worksheet expands on the material in that worksheet and also on the material introduced in worksheet 1.10. The distributive laws discussed in worksheet 1.10 are

$$\begin{aligned} a(b+c) &= ab+ac \\ (b+c)d &= bd+cd \end{aligned}$$

You may also recall that we showed the expansion of

$$(a+b)(x+y) = a(x+y)+b(x+y) = ax+ay+bx+by$$

We will now use these to expand expressions involving surds.

Example 1 :

$$\begin{aligned} (1+\sqrt{3})(1+\sqrt{3}) &= 1(1+\sqrt{3})+\sqrt{3}(1+\sqrt{3}) \\ &= 1+\sqrt{3}+\sqrt{3}+\sqrt{3}\times\sqrt{3} \\ &= 1+2\sqrt{3}+3 \\ &= 4+2\sqrt{3} \end{aligned}$$

Note: You should recall from worksheet 1.8 that

$$\sqrt{3}\times\sqrt{3}=3$$

and more generally,

$$\sqrt{a}\times\sqrt{b}=\sqrt{ab}$$

Example 2 :

$$\begin{aligned} (1+\sqrt{5})(2+\sqrt{3}) &= 1(2+\sqrt{3})+\sqrt{5}(2+\sqrt{3}) \\ &= 2+\sqrt{3}+2\sqrt{5}+\sqrt{15} \end{aligned}$$

Example 3 :

$$\begin{aligned} \sqrt{2}(5+\sqrt{8}) &= 5\sqrt{2}+\sqrt{16} \\ &= 5\sqrt{2}+4 \end{aligned}$$

Note: After expansion of this expression we ended up with a perfect square inside a square root sign. This was simplified. In a similar way surds that have perfect squares as factors should be simplified as far as possible. For example,

$$\begin{aligned}\sqrt{20} &= \sqrt{4}\times\sqrt{5} = 2\sqrt{5}\\ \sqrt{75} &= \sqrt{25}\times\sqrt{3} = 5\sqrt{3}\\ \sqrt{32} &= \sqrt{16}\times\sqrt{2} = 4\sqrt{2}\end{aligned}$$

Exercises:

1. Simplify the following surds:

   (a) $\sqrt{12}$ (b) $\sqrt{125}$ (c) $\sqrt{48}$ (d) $\sqrt{72}$ (e) $\sqrt{27}$

2. Expand and simplify the following expressions:

   (a) $\sqrt{2}(3+\sqrt{5})$
   (b) $\sqrt{6}(\sqrt{2}+\sqrt{8})$
   (c) $4(\sqrt{5}+3)$
   (d) $(2+\sqrt{3})(1+\sqrt{3})$
   (e) $(3-\sqrt{5})(3-2\sqrt{5})$
   (f) $(5-\sqrt{2})(5+\sqrt{2})$
   (g) $(2+\sqrt{5})(2+\sqrt{3})$
   (h) $(1-\sqrt{2})(1+\sqrt{3})$
   (i) $(8-\sqrt{2})(8+\sqrt{2})$
   (j) $(\sqrt{3}+\sqrt{5})(\sqrt{3}+\sqrt{5})$

Section 2 Fractions involving surds

As in the last worksheet on algebraic fractions, fractions involving surds are worked out similarly to fractions involving numbers. When adding and subtracting fractions the denominators must be the same for all the fractions involved in the calculation. This will normally involve finding equivalent fractions with the right denominator.

Example 1 :

$$\begin{aligned}\frac{2}{\sqrt{2}+\sqrt{8}}+\frac{1}{3} &= \frac{3\times 2}{3(\sqrt{2}+\sqrt{8})}+\frac{\sqrt{2}+\sqrt{8}}{3(\sqrt{2}+\sqrt{8})}\\ &= \frac{6+\sqrt{2}+\sqrt{8}}{3(\sqrt{2}+\sqrt{8})}\end{aligned}$$

Being able to factorize expressions involving surds often makes the expression much tidier. For this example we can get

$$\begin{aligned}
\frac{6+\sqrt{2}+\sqrt{8}}{3(\sqrt{2}+\sqrt{8})} &= \frac{6+\sqrt{2}(1+\sqrt{4})}{3\sqrt{2}(1+\sqrt{4})} \\
&= \frac{6+3\sqrt{2}}{9\sqrt{2}} \\
&= \frac{3(2+\sqrt{2})}{9\sqrt{2}} \\
&= \frac{2+\sqrt{2}}{3\sqrt{2}} \\
&= \frac{\sqrt{2}(\sqrt{2}+1)}{3\sqrt{2}} \\
&= \frac{\sqrt{2}+1}{3}
\end{aligned}$$

which is much tidier than the previous expression. We could have saved time and effort by simplifying the expression first. We needed to note that 8 has a factor which is a perfect square so

$$\sqrt{8} = \sqrt{2}\times\sqrt{4} = 2\sqrt{2}$$

and our initial sum becomes

$$\begin{aligned}
\frac{2}{\sqrt{2}+\sqrt{8}}+\frac{1}{3} &= \frac{2}{\sqrt{2}+2\sqrt{2}}+\frac{1}{3} \\
&= \frac{2}{3\sqrt{2}}+\frac{1}{3} \\
&= \frac{2}{3\sqrt{2}}+\frac{\sqrt{2}}{3\sqrt{2}} \\
&= \frac{2+\sqrt{2}}{3\sqrt{2}} \\
&= \frac{\sqrt{2}+1}{3}
\end{aligned}$$

Example 2 :

$$\frac{\sqrt{7}+1}{3}+\frac{\sqrt{7}-2}{4} = \frac{4(\sqrt{7}+1)}{3\times 4}+\frac{3(\sqrt{7}-2)}{4\times 3}$$
$$= \frac{4\sqrt{7}+4}{12}+\frac{3\sqrt{7}-6}{12}$$
$$= \frac{4\sqrt{7}+4+3\sqrt{7}-6}{12}$$
$$= \frac{7\sqrt{7}-2}{12}$$

Example 3 :

$$\frac{\sqrt{2}-3}{\sqrt{5}}-\frac{\sqrt{2}+4}{2} = \frac{2(\sqrt{2}-3)}{2\sqrt{5}}-\frac{\sqrt{5}(\sqrt{2}+4)}{2\sqrt{5}}$$
$$= \frac{2\sqrt{2}-6-\sqrt{10}-4\sqrt{5}}{2\sqrt{5}}$$

The last expression could be manipulated a little more, if desired, by noting that $\sqrt{10}=\sqrt{2}\sqrt{5}$.

Exercises:

1. Simplify the following:

   (a) $\frac{\sqrt{2}}{4}+\frac{\sqrt{2}+1}{5}$

   (b) $\frac{\sqrt{3}+2}{5}+\frac{\sqrt{3}+5}{7}$

   (c) $\frac{\sqrt{6}-1}{\sqrt{3}}+\frac{\sqrt{6}+2}{2\sqrt{3}}$

   (d) $\frac{\sqrt{5}-8}{4\sqrt{2}}-\frac{\sqrt{5}-3}{5\sqrt{2}}$

   (e) $\frac{4}{\sqrt{2}+1}+\frac{3}{\sqrt{2}+5}$

   (f) $\frac{4}{\sqrt{3}+\sqrt{7}}+\frac{5}{\sqrt{3}+2\sqrt{7}}$

   (g) $\frac{8+\sqrt{5}}{2}-\frac{4-2\sqrt{5}}{3}$

   (h) $\frac{2(\sqrt{5}+1)}{6}-\frac{3(\sqrt{5}+3)}{21}$

   (i) $\frac{4}{\sqrt{2}-1}+\frac{4+\sqrt{3}}{\sqrt{2}-3}$

   (j) $\frac{3(\sqrt{2}-7)}{\sqrt{5}}-\frac{5(\sqrt{2}+1)}{2}$

Section 3 Rationalizing the denominator

'Rationalizing the denominator' means to get all the fractional powers out of the denominator of a fraction. After rationalizing there should only be whole numbers on the bottom of the fraction and no surds. In effect what we want to do is find an equivalent fraction. You already know that to find an equivalent fraction you need to multiply the top and bottom of the fraction by the same number or expression, which effectively multiplies by 1. Therefore to rationalize the denominator we need to find an expression which, when multiplied with an expression containing surds, gives only fractions or whole numbers. To achieve this, it is important to notice that

$$(a-b)(a+b) = a(a+b) - b(a+b) = a^2 - b^2$$

Note: $a^2 - b^2$ is called the difference of squares.

The importance of the difference of squares formula is that if a and b are both surds (for example $a = \sqrt{2}$, $b = \sqrt{3}$), then the expression $a^2 - b^2$ contains *no* surds.

Example 1 :

$$\begin{aligned}(1+\sqrt{2})(1-\sqrt{2}) &= 1(1-\sqrt{2}) + \sqrt{2}(1-\sqrt{2}) \\ &= 1 - \sqrt{2} + \sqrt{2} - \sqrt{2} \times \sqrt{2} \\ &= 1 - 2 \\ &= -1\end{aligned}$$

Example 2 :

$$\begin{aligned}(3+\sqrt{5})(3-\sqrt{5}) &= 3(3-\sqrt{5}) + \sqrt{5}(3-\sqrt{5}) \\ &= 9 - 3\sqrt{5} + 3\sqrt{5} - 5 \\ &= 4\end{aligned}$$

We can use this information to help us rationalize the denominator of fractions with expressions containing square roots in the denominator.

Example 3 :

$$\frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Example 4 :

$$\begin{aligned}\frac{5}{3\sqrt{7}} &= \frac{5}{3\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} \\ &= \frac{5\sqrt{7}}{21}\end{aligned}$$

Example 5 :

$$\begin{aligned}\frac{6}{5\sqrt{2}} &= \frac{6}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{6\sqrt{2}}{10} \\ &= \frac{3\sqrt{2}}{5}\end{aligned}$$

Example 6 :

$$\begin{aligned}\frac{\sqrt{3}+1}{\sqrt{3}-4} &= \frac{\sqrt{3}+1}{\sqrt{3}-4} \times \frac{\sqrt{3}+4}{\sqrt{3}+4} \\ &= \frac{\sqrt{3}(\sqrt{3}+4)+1(\sqrt{3}+4)}{3-16} \\ &= \frac{3+4\sqrt{3}+\sqrt{3}+4}{-13} \\ &= \frac{7+5\sqrt{3}}{-13} \\ &= -\frac{7+5\sqrt{3}}{13}\end{aligned}$$

Example 7 :

$$\begin{aligned}\frac{1}{1+\sqrt{3}} &= \frac{1}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}} \\ &= \frac{1-\sqrt{3}}{1-3} \\ &= \frac{\sqrt{3}-1}{2}\end{aligned}$$

Example 8 :

$$\begin{aligned}\frac{1}{3-\sqrt{5}} &= \frac{1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}} \\ &= \frac{3+\sqrt{5}}{9-5} \\ &= \frac{3+\sqrt{5}}{4}\end{aligned}$$

Example 9 :

$$\begin{aligned}\frac{\sqrt{2}}{6+\sqrt{2}} &= \frac{\sqrt{2}}{6+\sqrt{2}} \times \frac{6-\sqrt{2}}{6-\sqrt{2}} \\ &= \frac{\sqrt{2}(6-\sqrt{2})}{36-2} \\ &= \frac{6\sqrt{2}-2}{34} \\ &= \frac{3\sqrt{2}-1}{17}\end{aligned}$$

Exercises:

1. Rewrite the following expressions with rational denominators:

   (a) $\frac{3}{\sqrt{5}}$
   (b) $\frac{4}{\sqrt{8}}$
   (c) $\frac{9}{\sqrt{48}}$
   (d) $\frac{\sqrt{2}+1}{\sqrt{2}}$
   (e) $\frac{\sqrt{3}-1}{\sqrt{5}}$
   (f) $-\frac{4}{3\sqrt{2}}$
   (g) $\frac{\sqrt{5}+3}{\sqrt{10}}$
   (h) $\frac{\sqrt{2}-1}{\sqrt{7}}$
   (i) $\frac{1}{\sqrt{3}-1}$
   (j) $\frac{4}{\sqrt{6}-2}$
   (k) $\frac{7}{\sqrt{7}-2}$
   (l) $\frac{-3}{\sqrt{5}+1}$
   (m) $\frac{\sqrt{2}+3}{\sqrt{5}}$
   (n) $\frac{\sqrt{5}-1}{\sqrt{5}+3}$
   (o) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+4}$
   (p) $\frac{5+2\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

Exercises for Worksheet 2.5

1. (a) $\sqrt{2} \times \sqrt{2}$
   (b) $(\sqrt{5})^2$
   (c) $\sqrt{3} \times \sqrt{2}$
   (d) $6\sqrt{3} + 2\sqrt{3}$
   (e) $2(3 - \sqrt{7})$
   (f) $(2 + \sqrt{2}) - (3 + 2\sqrt{2})$
   (g) $3\sqrt{3} - 2(\sqrt{3} + 2)$
   (h) $\sqrt{8}$
   (i) $\sqrt{a^2}$
   (j) $\sqrt{49b^4}$
   (k) $\frac{1}{p}\sqrt{\frac{p^4}{16}}$
   (l) $5\sqrt{\frac{25}{4}}$

2. Simplify:
   (a) $5\sqrt{12} + \sqrt{27}$
   (b) $3\sqrt{20} - 3\sqrt{80} - 2\sqrt{45}$
   (c) $\sqrt{15} \times \sqrt{3}$
   (d) $2\sqrt{20} \times \frac{\sqrt{45}}{3}$
   (e) $(2 + \sqrt{3})(2 - \sqrt{3})$
   (f) $(3\sqrt{2} + 2\sqrt{3})^2$
   (g) $(3\sqrt{5} - 2)(\sqrt{5} + 3)$
   (h) $(2\sqrt{5} - \sqrt{3})(3\sqrt{3} + \sqrt{5})$
   (i) $\frac{2}{\sqrt{3}}$
   (j) $\frac{2}{\sqrt{5}+1}$
   (k) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$
   (l) $\frac{5}{\sqrt{2}+\sqrt{8}} - \frac{1}{4}$
   (m) $\frac{2}{\sqrt{3}+1} + \frac{4}{\sqrt{3}-2}$
   (n) $\frac{1}{\sqrt{2}+3} - \frac{3}{\sqrt{2}-1}$

3. (a) Find the value of $x^2 + 4x + 4$ when $x = 2 + \sqrt{3}$.
   (b) Find the value of $2x^2 - 3xy$ when $x = \sqrt{2} + 3$ and $y = \sqrt{2} - 2$.
   (c) Given $\sqrt{2} = 1.41$ (to 2 decimal places), simplify $\frac{1}{\sqrt{2}}$ without a calculator. (Rationalize the denominator.)
   (d) Given $\sqrt{3} = 1.73$ (to 2 decimal places), simplify $\frac{3}{2+\sqrt{3}}$ without a calculator.
   (e) Find the area of a circle of radius $2\sqrt{7}$ cm (correct to 2 decimal places).
   (f) Find the perimeter of a rectangle of length $(3 + \sqrt{2})$ and breadth $(\sqrt{2} - 1)$ cm.

Worksheet 2.6 Factorizing Algebraic Expressions

Section 1 Finding Factors

Factorizing algebraic expressions is a way of turning a sum of terms into a product of smaller ones. The product is a multiplication of the factors. Sometimes it helps to look at a simpler case before venturing into the abstract. The number 48 may be written as a product in a number of different ways:

$$48 = 3 \times 16 = 4 \times 12 = 2 \times 24$$

So too can polynomials, unless of course the polynomial has no factors (in the way that the number 23 has no factors). For example:

$$x^3 - 6x^2 + 12x - 8 = (x-2)^3 = (x-2)(x-2)(x-2) = (x-2)(x^2 - 4x + 4)$$

where $(x-2)^3$ is in fully factored form.

Occasionally we can start by taking common factors out of every term in the sum. For example,

$$\begin{aligned} 3xy + 9xy^2 + 6x^2y &= 3xy(1) + 3xy(3y) + 3xy(2x) \\ &= 3xy(1 + 3y + 2x) \end{aligned}$$

Sometimes not all the terms in an expression have a common factor but you may still be able to do some factoring.

Example 1 :

$$9a^2b + 3a^2 + 5b + 5b^2a = 3a^2(3b+1) + 5b(1+ba)$$

Example 2 :

$$\begin{aligned} 10x^2 + 5x + 2xy + y &= 5x(2x+1) + y(2x+1) \qquad \text{Let } T = 2x+1 \\ &= 5xT + yT \\ &= T(5x+y) \\ &= (2x+1)(5x+y) \end{aligned}$$

Example 3 :

$$\begin{aligned} x^2 + 2xy + 5x^3 + 10x^2y &= x(x+2y) + 5x^2(x+2y) \\ &= (x+5x^2)(x+2y) \\ &= x(1+5x)(x+2y) \end{aligned}$$

Exercises:

1. Factorize the following algebraic expressions:

   (a) $6x + 24$

   (b) $8x^2 - 4x$

   (c) $6xy + 10x^2y$

   (d) $m^4 - 3m^2$

   (e) $6x^2 + 8x + 12yx$

   For the following expressions, factorize the first pair, then the second pair:

   (f) $8m^2 - 12m + 10m - 15$

   (g) $x^2 + 5x + 2x + 10$

   (h) $m^2 - 4m + 3m - 12$

   (i) $2t^2 - 4t + t - 2$

   (j) $6y^2 - 15y + 4y - 10$

Section 2 Some standard factorizations

Recall the distributive laws of section 1.10.

Example 1 :

$$\begin{aligned}(x+3)(x-3) &= x(x-3) + 3(x-3)\\ &= x^2 - 3x + 3x - 9\\ &= x^2 - 9\\ &= x^2 - 3^2\end{aligned}$$

Example 2 :

$$\begin{aligned}(x+9)(x-9) &= x(x-9) + 9(x-9)\\ &= x^2 - 9x + 9x - 81\\ &= x^2 - 81\\ &= x^2 - 9^2\end{aligned}$$

Notice that in each of these examples, we end up with a quantity in the form $A^2 - B^2$. In example 1, we have

$$\begin{aligned} A^2 - B^2 &= x^2 - 9 \\ &= (x+3)(x-3) \end{aligned}$$

where we have identified $A = x$ and $B = 3$. In example 2, we have

$$\begin{aligned} A^2 - B^2 &= x^2 - 81 \\ &= (x+9)(x-9) \end{aligned}$$

where we have identified $A = x$ and $B = 9$. The result that we have developed and have used in two examples is called the difference of two squares, and is written:

$$A^2 - B^2 = (A+B)(A-B)$$

The next common factorization that is important is called a perfect square. Notice that

$$\begin{aligned} (x+5)^2 &= (x+5)(x+5) \\ &= x(x+5) + 5(x+5) \\ &= x^2 + 5x + 5x + 25 \\ &= x^2 + 10x + 25 \\ &= x^2 + 2(5x) + 5^2 \end{aligned}$$

The perfect square is written as:

$$(x+a)^2 = x^2 + 2ax + a^2$$

Similarly,

$$\begin{aligned} (x-a)^2 &= (x-a)(x-a) \\ &= x(x-a) - a(x-a) \\ &= x^2 - ax - ax + a^2 \\ &= x^2 - 2ax + a^2 \end{aligned}$$

For example,

$$\begin{aligned} (x-7)^2 &= (x-7)(x-7) \\ &= x(x-7) - 7(x-7) \\ &= x^2 - 7x - 7x + 7^2 \\ &= x^2 - 14x + 49 \end{aligned}$$

Exercises:

1. Expand the following, and collect like terms:
   (a) $(x+2)(x-2)$
   (b) $(y+5)(y-5)$
   (c) $(y-6)(y+6)$
   (d) $(x+7)(x-7)$
   (e) $(2x+1)(2x-1)$
   (f) $(3m+4)(3m-4)$
   (g) $(3y+5)(3y-5)$
   (h) $(2t+7)(2t-7)$
2. Factorize the following:
   (a) x^2-16
   (b) y^2-49
   (c) x^2-25
   (d) $4x^2-25$
   (e) $16-y^2$
   (f) m^2-36
   (g) $4m^2-49$
   (h) $9m^2-16$
3. Expand the following and collect like terms:
   (a) $(x+5)(x+5)$
   (b) $(x+9)(x+9)$
   (c) $(y-2)(y-2)$
   (d) $(m-3)(m-3)$
   (e) $(2m+5)(2m+5)$
   (f) $(t+10)(t+10)$
   (g) $(y+8)^2$
   (h) $(t+6)^2$
4. Factorize the following:
   (a) y^2-6y+9
   (b) $x^2-10x+25$
   (c) $x^2+8x+16$
   (d) $x^2+20x+100$
   (e) $m^2+16m+64$
   (f) $t^2-30t+225$
   (g) $m^2-12m+36$
   (h) $t^2+18t+81$

Section 3 Introduction to Quadratics

In the expression $5t^2 + 2t + 1$, t is called the variable. Quadratics are algebraic expressions of one variable, and they have degree two. Having degree two means that the highest power of the variable that occurs is a squared term. The general form for a quadratic is

$$ax^2 + bx + c$$

Note that we assume that a is not zero because if it were zero, we would have $bx + c$ which is not a quadratic: the highest power of x would not be two, but one. There are a few points to make about the quadratic $ax^2 + bx + c$:

1. a is the coefficient of the squared term and $a \neq 0$.
2. b is the coefficient of x and can be any number.
3. c is the called the constant term (even though a and b are also constant), and can be any number.

Quadratics may factor into two linear factors:

$$ax^2 + bx + c = a(x + k)(x + l)$$

where $(x + k)$ and $(x + l)$ are called the linear factors.

Exercises:

1. Which of the following algebraic expressions is a quadratic?

   (a) $x^2 - 3x + 4$
   (b) $4x^2 + 6x - 1$
   (c) $x^3 - 6x + 2$
   (d) $\frac{1}{x^2} + 2x + 1$
   (e) $x^2 - 4$
   (f) $6x^2$

Section 4 Factorizing Quadratics

Before we start factorizing quadratics, it would be a good idea to look for a pattern.

$$\begin{aligned}(x+2)(x+4) &= x^2+4x+2x+8 \\ &= x^2+6x+8\end{aligned}$$

Notice that the numbers 2 and 4 add to give 6 and multiply to give 8.

$$\begin{aligned}(x+5)(x-3) &= x^2-3x+5x-15 \\ &= x^2+2x-15\end{aligned}$$

Notice that the numbers 5 and -3 add to give 2 and multiply to give -15.

Let's try to factorize expressions similar to those above, where we will start with the expression in its expanded out form. To factorize the expression $x^2+7x+12$, we will try to find numbers that multiply to give 12 and add to give 7. The numbers that we come up with are 3 and 4, so we write

$$x^2+7x+12 = (x+3)(x+4)$$

This equation should be verified by expanding the right hand side.

Example 1 : Factorize $x^2+9x+14$.

We attempt to find two numbers that add to give 9 and multiply to give 14, and the numbers that do this are 2 and 7. Therefore

$$x^2+9x+14 = (x+2)(x+7)$$

Again, this equation shouldn't be believed until the right hand side is expanded, and is shown to equal $x^2+9x+14$.

Example 2 : Factorize $x^2+7x-18$.

We attempt to find two numbers that add to give 7 and multiply to give -18 (notice the minus!). The numbers that do this are -2 and 9. Therefore

$$x^2+7x-18 = (x-2)(x+9)$$

This equation shouldn't be believed until the right hand side is expanded, and is shown to equal $x^2+7x-18$.

Exercises:

1. Factorize the following quadratics:

   (a) $x^2 + 4x + 3$

   (b) $x^2 + 15x + 44$

   (c) $x^2 + 11x - 26$

   (d) $x^2 + 7x - 30$

   (e) $x^2 + 10x + 24$

   (f) $x^2 - 14x + 24$

   (g) $x^2 - 7x + 10$

   (h) $x^2 - 5x - 24$

   (i) $x^2 + 2x - 15$

   (j) $x^2 - 2x - 15$

The method that we have just described to factorize quadratics will work, if at all, only in the case that the coefficient of x^2 is 1. For other cases, we will need to factorize by

1. Using the 'ACE' method, or by
2. Using the quadratic formula

The 'ACE' method (pronounced a-c), unlike some other methods, is clear and easy to follow, as each step leads logically to the next. If you can expand an expression like $(3x + 4)(2x - 3)$, then you will be able to follow this technique.

Example	Factorize $6x^2 - x - 12$
1: Multiply the first term $6x^2$ by the last term (-12) 2: Find factors of $-72x^2$ which add to $-x$. 3: Return to the original expression and replace $-x$ with $-9x + 8x$. 4: Factorize $(6x^2 - 9x)$ and $(8x - 12)$. 5: One common factor is $(2x - 3)$. The other factor, $(3x+4)$, is found by dividing each term by $(2x - 3)$.	$-72x^2$ $(-9x)(8x) = -72x^2$ $-9x + 8x = -x$ $6x^2 - x - 12$ $= 6x^2 - 9x + 8x - 12$ $= 3x(2x - 3) + 4(2x - 3)$ $= (2x - 3)(3x + 4)$
6: Verify the factorization by expansion	$(3x + 4)(2x - 3)$ $= 3x(2x - 3) + 4(2x - 3)$ $= 6x^2 - 9x + 8x - 12$ $= 6x^2 - x - 12$

Example 3 : Factorize $4x^2 + 21x + 5$.

1. Multiply first and last terms: $4x^2 \times 5 = 20x^2$
2. Find factors of $20x^2$ which add to $21x$ and multiply to give $20x^2$. These are $20x$ and x.
3. Replace $21x$ in the original expression with $20x + x$:

$$4x^2 + 21x + 5 = 4x^2 + 20x + x + 5$$

4. Factorize the first two terms and the last two terms

$$4x^2 + 20x + x + 5 = 4x(x + 5) + (x + 5)$$

5. Factorize further:

$$4x(x+5)+(x+5)=(x+5)(4x+1)$$

Exercises:

1. Factorize the following quadratics using the 'ACE' method:

(a) $2x^2+11x+12$	(f) $2x^2-5x-3$
(b) $3x^2+16x+5$	(g) $3x^2-10x-8$
(c) $6x^2+17x+12$	(h) $3x^2-11x-20$
(d) $2x^2+9x+10$	(i) $5x^2+17x+6$
(e) $12x^2+11x+2$	(j) $10x^2+19x+6$

Section 5 The quadratic formula

When there is no obvious whole-number solution to the quadratic factorization, the quadratic formula must be used. It can be shown by the method of completing the square that the solutions to $ax^2+bx+c=0$ are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

If we let the roots be k and l, say, then

$$\begin{aligned} k &= \frac{-b+\sqrt{b^2-4ac}}{2a} \\ l &= \frac{-b-\sqrt{b^2-4ac}}{2a} \end{aligned}$$

Then

$$ax^2+bx+c=a(x-k)(x-l)$$

When factorizing using this method be sure to multiply throughout by the coefficient of x^2.

Example 1 : Factorize x^2+5x+3. We try the 'ACE' method. The obvious factors of $3x^2$ are $3x$ and $1x$, which won't add to $5x$. We abandon this method, and go to the quadratic formula. Note that $a = 1$, $b = 5$, and $c = 3$.

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)} \\ &= -\frac{5}{2} \pm \frac{\sqrt{13}}{2} \end{aligned}$$

so that the two roots are

$$k_1 = \frac{-5 + \sqrt{13}}{2} \quad \text{and} \quad k_2 = \frac{-5 - \sqrt{13}}{2}$$

Then

$$x^2 + 5x + 3 = (x - \frac{-5 + \sqrt{13}}{2})(x - \frac{-5 - \sqrt{13}}{2})$$

Example 2 : Factorize $2x^2 - x - 5$.

Note that $a = 2$, $b = -1$, and $c = -5$. Then the solutions to $2x^2 - x - 5 = 0$ are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{1 \pm \sqrt{(-1)^2 - 4 \times 2 \times (-5)}}{2 \times 2} \\ &= \frac{1 \pm \sqrt{41}}{4} \end{aligned}$$

So the two factors of $2x^2 - x - 5$ are

$$(x - \frac{1 + \sqrt{41}}{4}) \qquad \text{and} \qquad (x - \frac{1 - \sqrt{41}}{4})$$

and so the factorization is

$$2x^2 - x - 5 = 2\left(x - \frac{1 + \sqrt{41}}{4}\right)\left(x - \frac{1 - \sqrt{41}}{4}\right)$$

This right hand side of this equation should be expanded before it is believed!

Exercises:

1. Factorize the following quadratics using the quadratic formula:

(a) $3x^2 + 2x - 4$

(b) $x^2 + 3x + 1$

(c) $2x^2 + 8x + 3$

(d) $3x^2 + 5x + 1$

(e) $3x^2 + 6x + 2$

(f) $5x^2 + 7x - 2$

(g) $3x^2 + 5x - 4$

(h) $2x^2 + 4x + 1$

(i) $5x^2 + 2x - 2$

(j) $2x^2 + x - 7$

Section 6 Uses of factorization

We can use factorization of expressions in a variety of ways. One way is to simplify algebraic fractions.

Example 1 :

$$\begin{aligned} \frac{x^2-9}{x-3} &= \frac{(x-3)(x+3)}{(x-3)} \\ &= \frac{x-3}{x-3} \times (x+3) \\ &= x+3 \end{aligned}$$

Example 2 :

$$\begin{aligned} \frac{x}{x^2+4x+4} + \frac{x}{x+2} &= \frac{x}{(x+2)^2} + \frac{x}{x+2} \\ &= \frac{x}{(x+2)^2} + \frac{x}{x+2} \times \frac{x+2}{x+2} \\ &= \frac{x}{(x+2)^2} + \frac{x^2+2x}{(x+2)^2} \\ &= \frac{x^2+x+2x}{(x+2)^2} \\ &= \frac{x(x+3)}{(x+2)^2} \end{aligned}$$

Another way of using factorization is in solving quadratic equations.

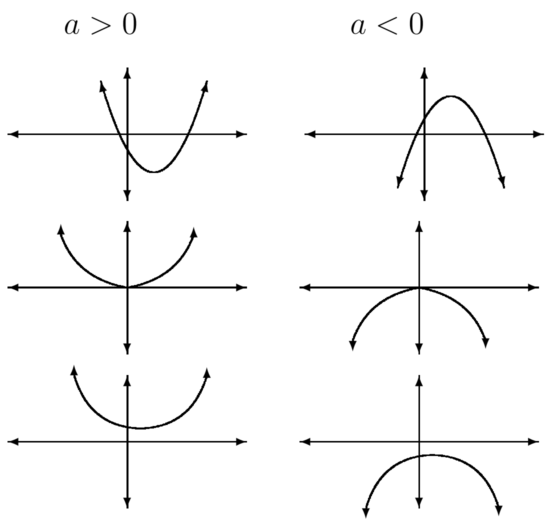
Example 3 : Solve $(x+3)^2 = x+5$.

$$\begin{aligned} x^2+6x+9 &= x+5 \\ x^2+5x+4 &= 0 \\ (x+4)(x+1) &= 0 \end{aligned}$$

This is true when either $x=-4$ or $x=-1$. In other words, just one of the factors needs to be zero for the original equation that we started with to be true.

It is a good idea to know what to expect from the equation by first examining the discriminant $\Delta = b^2 - 4ac$. This is the expression under the square-root sign in the quadratic formula. Given the equation $y = ax^2 + bx + c$, and using our knowledge of square roots, we find the following:

$\Delta > 0$	There will be 2 distinct solutions, so the curve crosses the x-axis twice.
$\Delta = 0$	There will only be one solution, so the curve will only touch the x-axis. This is called a double root.
$\Delta < 0$	The curve does not touch the x-axis. We will deal with this case in detail later.



Exercises:

1. Factorize and then simplify the following algebraic expressions:
   (a) $\frac{x^2+3x}{x+3}$
   (b) $\frac{6x^2-8}{2x}$
   (c) $\frac{x^2+3x+2}{3x+6}$
   (d) $\frac{x^2-7x-18}{x^2-6x-27}$
   (e) $\frac{x^2-16}{2x+8}$
   (f) $\frac{3x^2-9x}{18x}$

(g) $\frac{x^2-25}{x^2-3x-10}$

(h) $\frac{2x^2-32}{x^2+6x+8}$

(i) $\frac{x^3-9x^2}{3x-27}$

(j) $\frac{2x^2-x-6}{x^2+x-6}$

2. Simplify the following by first finding a common denominator:

(a) $\frac{3}{x+2} + \frac{5x}{x+3}$

(b) $\frac{4x}{x-5} - \frac{2}{x+2}$

(c) $\frac{x+1}{x+2} + \frac{x+3}{x+4}$

(d) $\frac{6}{x^2+5x+6} + \frac{2}{x^2+8x+15}$

(e) $\frac{4}{x^2-3x-10} - \frac{1}{x^2+5x+6}$

(f) $\frac{x+3}{x^2+6x+9} - \frac{2}{x+3}$

(g) $\frac{x^2+8x+15}{x^2+7x+10} - \frac{x+3}{x+2}$

(h) $\frac{x^2-9}{2x+6} - \frac{x^2}{x-3}$

(i) $\frac{x}{x+1} - \frac{2x}{x+3}$

(j) $\frac{3x}{x^2+6x} - \frac{2x+1}{x+6}$

3. Solve the following quadratic equations:

(a) $x^2 - 6x + 8 = 0$

(b) $x^2 + 8x + 15 = 0$

(c) $x^2 + 7x + 12 = 0$

(d) $x^2 + 9x - 22 = 0$

(e) $x^2 - 7x + 12 = 0$

(f) $2x^2 - x - 6 = 0$

(g) $2x^2 - 13x - 7 = 0$

(h) $3x^2 - 10x - 8 = 0$

(i) $7x^2 + 13x - 2 = 0$

(j) $x^2 - 18x + 77 = 0$

4. Solve the following equations using the quadratic formula. Write the answers to two decimal places.

(a) $x^2 - 3x + 1 = 0$

(b) $2x^2 - 6x - 7 = 0$

(c) $3x^2 + 2x - 2 = 0$

(d) $2x^2 - 13x + 7 = 0$

Section 7 Multiplication and Division of Algebraic Fractions

We are often able to use factorization when we are multiplying or dividing algebraic expressions.

Example 1 :

$$\begin{aligned}\frac{x^2-16}{x+3}\times\frac{x^2+5x+6}{x+4} &= \frac{(x+4)(x-4)}{x+3}\times\frac{(x+3)(x+2)}{x+4}\\ &= (x-4)(x+2)\end{aligned}$$

Example 2 :

$$\begin{aligned}\frac{2x^2+12x+16}{3x^2+6x}\times\frac{4x^2-100}{6x+30} &= \frac{2(x^2+6x+8)}{3x(x+2)}\times\frac{4(x^2-25)}{6(x+5)}\\ &= \frac{2(x+4)(x+2)}{3x(x+2)}\times\frac{4(x+5)(x-5)}{6(x+5)}\\ &= \frac{4(x+4)(x-5)}{9x}\end{aligned}$$

Example 3 :

$$\begin{aligned}\frac{6x^2+9x}{x^2+8x+15}\div\frac{4x+6}{x^2-9} &= \frac{6x^2+9x}{x^2+8x+15}\times\frac{x^2-9}{4x+6}\\ &= \frac{3x(2x+3)}{(x+3)(x+5)}\times\frac{(x+3)(x-3)}{2(2x+3)}\\ &= \frac{3x(x-3)}{2(x+5)}\end{aligned}$$

Exercises:

1. Simplify the following expressions:
   (a) $\frac{2x^2-5x-3}{x^2+2x}\times\frac{x^2+4x}{2x+1}$
   (b) $\frac{3x+21}{x^2-7x+12}\times\frac{4x-12}{9x+63}$
   (c) $\frac{x^2+2x}{x^2+x-20}\times\frac{2x^2-5x-12}{5x+10}$
   (d) $\frac{3x^2-10x-8}{5x-15}\div\frac{2x^2-7x-4}{x^2-3x}$
   (e) $\frac{x^2-16}{4x^2-1}\div\frac{x^2+11x+28}{2x^2+5x+2}$

Exercises for Worksheet 2.6

1. Expand

   (a) $(x+2)(x-2)$

   (b) $(2x+4y)(2x-4y)$

2. Factorize

   (a) $144x^2 - y^2$

   (b) $16a^2 - 9b^2$

   (c) $x^2 + 7x + 10$

   (d) $b^2 + 9b + 14$

   (e) $x^2 + 14x + 49$

   (f) $x^2 + x - 6$

   (g) $2x^3 + 10x^2 - 48x$

   (h) $2x^2 + 7x + 3$

   (i) $5b^2 + 17b + 6$

   (j) $18y^2 + 12y + 2$

   (k) $12x^2 + 6x - 6$

   (l) $7a^2 - 9a - 10$

   (m) $-x^2 - x + 2$

   (n) $-2x^2 + 3x + 2$

3. (a) Factorize $x^2 + 5x + 3$ using the quadratic formula.

   (b) Factorize $2x^2 - x - 1$.

   (c) Simplify $\frac{A^2-4}{3A-6}$.

   (d) Simplify $\frac{2x^2-8}{2x^2-x-6}$.

   (e) Simplify $\frac{x^2-x-6}{2xy} \times \frac{2x^2y}{x^2-9}$.

   (f) Simplify $\frac{2x^2+5x-3}{x^3+3x^2+2x} \div \frac{4x^2-1}{x^3+2x^2}$.

   (g) Simplify $\frac{3x}{x^2+6x+9} + \frac{x+3}{x^2-9}$.

   (h) Solve $x^2 + 2x - 3 = 0$.

Worksheet 2.7 Logarithms and Exponentials

Section 1 LOGARITHMS

The mathematics of logarithms and exponentials occurs naturally in many branches of science. It is very important in solving problems related to growth and decay. The growth and decay may be that of a plant or a population, a crystalline structure or money in the bank. Therefore we need to have some understanding of the way in which logs and exponentials work.

Definition: If x and b are positive numbers and $b \neq 1$ then the logarithm of x to the base b is the power to which b must be raised to equal x. It is written $\log_b x$. In algebraic terms this means that

$$\begin{aligned} \text{if } y &= \log_b x \quad \text{then} \\ x &= b^y \end{aligned}$$

The formula $y = \log_b x$ is said to be written in logarithmic form and $x = b^y$ is said to be written in exponential form. In working with these problems it is most important to remember that $y = \log_b x$ and $x = b^y$ are equivalent statements.

Example 1 : If $\log_4 x = 2$ then

$$\begin{aligned} x &= 4^2 \\ x &= 16 \end{aligned}$$

Example 2 : We have $25 = 5^2$. Then $\log_5 25 = 2$.

Example 3 : If $\log_9 x = \frac{1}{2}$ then

$$\begin{aligned} x &= 9^{\frac{1}{2}} \\ x &= \sqrt{9} \\ x &= 3 \end{aligned}$$

Example 4 : If $\log_2 \frac{y}{3} = 4$ then

$$\begin{aligned} \frac{y}{3} &= 2^4 \\ \frac{y}{3} &= 16 \\ y &= 16 \times 3 \\ y &= 48 \end{aligned}$$

Exercises:

1. Write the following in exponential form:
   (a) $\log_3 x = 9$
   (b) $\log_2 8 = x$
   (c) $\log_3 27 = x$
   (d) $\log_4 x = 3$
   (e) $\log_2 y = 5$
   (f) $\log_5 y = 2$
2. Write the following in logarithm form:
   (a) $y = 3^4$
   (b) $27 = 3^x$
   (c) $m = 4^2$
   (d) $y = 3^5$
   (e) $32 = x^5$
   (f) $64 = 4^x$
3. Solve the following:
   (a) $\log_3 x = 4$
   (b) $\log_m 81 = 4$
   (c) $\log_x 1000 = 3$
   (d) $\log_2 \frac{x}{2} = 5$
   (e) $\log_3 y = 5$
   (f) $\log_2 4x = 5$

Section 2 Properties of Logs

Logs have some very useful properties which follow from their definition and the equivalence of the logarithmic form and exponential form. Some useful properties are as follows:

$$\begin{aligned} \log_b mn &= \log_b m + \log_b n \\ \log_b \frac{m}{n} &= \log_b m - \log_b n \\ \log_b m^a &= a \log_b m \\ \log_b m &= \log_b n \quad \text{if and only if} \quad m = n \end{aligned}$$

Note that for all of the above properties we require that $b > 0$, $b \neq 1$, and $m, n > 0$. Note also that $\log_b 1 = 0$ for any $b \neq 0$ since $b^0 = 1$. In addition, $\log_b b = 1$ since $b^1 = b$. We can apply these properties to simplify logarithmic expressions.

Example 1 :

$$\begin{aligned} \log_b \frac{xy}{z} &= \log_b xy - \log_b z \\ &= \log_b x + \log_b y - \log_b z \end{aligned}$$

Example 2 :

$$\begin{aligned} \log_5 5^p &= p \log_5 5 \\ &= p \times 1 \\ &= p \end{aligned}$$

Example 3 :

$$\begin{aligned} \log_2 (8x)^{\frac{1}{3}} &= \frac{1}{3} \log_2 8x \\ &= \frac{1}{3}[\log_2 8 + \log_2 x] \\ &= \frac{1}{3}[3 + \log_2 x] \\ &= 1 + \frac{1}{3} \log_2 x \end{aligned}$$

Example 4 : Find x if

$$2 \log_b 5 + \frac{1}{2} \log_b 9 - \log_b 3 = \log_b x$$

$$\begin{aligned} \log_b 5^2 + \log_b 9^{\frac{1}{2}} - log_b 3 &= \log_b x \\ \log_b 25 + \log_b 3 - \log_b 3 &= \log_b x \\ \log_b 25 &= \log_b x \\ x &= 25 \end{aligned}$$

Example 5 :

$$\begin{aligned}\log_2 \frac{8x^3}{2y} &= \log_2 8x^3 - \log_2 2y \\ &= \log_2 8 + \log_2 x^3 - [\log_2 2 + \log_2 y] \\ &= 3 + 3\log_2 x - [1 + \log_2 y] \\ &= 3 + 3\log_2 x - 1 - \log_2 y \\ &= 2 + 3\log_2 x - \log_2 y\end{aligned}$$

Exercises:

1. Use the logarithm laws to simplify the following:
   (a) $\log_2 xy - \log_2 x^2$
   (b) $\log_2 \frac{8x^2}{y} + \log_2 2xy$
   (c) $\log_3 9xy^2 - \log_3 27xy$
   (d) $\log_4 (xy)^3 - \log_4 xy$
   (e) $\log_3 9x^4 - \log_3 (3x)^2$
2. Find x if:
   (a) $2\log_b 4 + \log_b 5 - \log_b 10 = \log_b x$
   (b) $\log_b 30 - \log_b 5^2 = \log_b x$
   (c) $\log_b 8 + \log_b x^2 = \log_b x$
   (d) $\log_b (x + 2) - \log_b 4 = \log_b 3x$
   (e) $\log_b (x - 1) + \log_b 3 = \log_b x$

Section 3 The Natural Logarithm and Exponential

The natural logarithm is often written as ln which you may have noticed on your calculator.

$$\ln x = \log_e x$$

The symbol e symbolizes a special mathematical constant. It has importance in growth and decay problems. The logarithmic properties listed above hold for all bases of logs. If you see $\log x$ written (with no base), the natural log is implied. The number e can not be written exactly in decimal form, but it is approximately 2.718. Of course, all the properties of logs that we have written down also apply to the natural log. In particular,

$$e^y = x \quad \text{and} \quad \ln x = y$$

are equivalent statements. We also have $e^0 = 1$ and $\ln 1 = 0$.

<u>Example 1</u> : $e^{\log_e a} = a$

<u>Example 2</u> : $e^{a \log_e x} = e^{\log_e x^a} = x^a$

<u>Example 3</u> :

$$\begin{aligned} \log_e e^{2y} &= 2y \log_e e \\ &= 2y \end{aligned}$$

<u>Example 4</u> : $\log_e \frac{x^2}{5} = 2 \log_e x - \log_e 5$

Exercises:

1. Use your calculator to find the following:

   (a) $\ln 1.4$
   (b) $\ln 0.872$
   (c) $\ln \frac{6.4 \times 3.8}{10}$
   (d) $e^{0.62}$
   (e) $e^{3.8}$
   (f) $(e^{0.24})^2$
   (g) $e^{1.4} \times e^{0.8}$
   (h) $6e^{-4.1}$
   (i) $\frac{e^{8.2}}{1068}$
   (j) $e^{-2.4} \times e^{6.1} \div (8 + \ln 2)$

2. Simplify the following

   (a) $\log x^2 - \log xy + 4 \log y$
   (b) $\ln(8x)^{\frac{1}{2}} + \ln 4x^2 - \ln(16x)^{\frac{1}{2}}$
   (c) $e^6 e^{-6}$
   (d) $12e^7 \div 6e^2$
   (e) $\ln e^2$
   (f) $\ln(e^2 \ln e^3)$

3. Find x in each of the following:

   (a) $\ln x = 2.7$
   (b) $\ln(x + 1) = 1.86$
   (c) $x = e^{9.8} \div e^{7.6}$
   (d) $6.27 = e^{x}$
   (e) $4.12 = e^{-2x}$

Exercises for Worksheet 2.7

1. Evaluate

   (a) $\log_{10} 1000$

   (b) $\log_4 1$

   (c) $\log_3 27$

   (d) $\log_2 \frac{1}{4}$

   (e) $\log_a a^x$

2. Solve for x

   (a) $\log_4 x = 2$

   (b) $\log_{\frac{1}{3}} x = 4$

   (c) $\log_{10}(2x + 1) = 2$

   (d) $\log_2 64 = x$

   (e) $\log_b 81 = 4$

3. (a) Use log laws to solve $\log_3 x = \log_3 7 + \log_3 3$.

   (b) Without tables, simplify $2 \log_{10} 5 + \log_{10} 8 - \log_{10} 2$.

   (c) If $\log_{10} 8 = x$ and $\log_{10} 3 = y$, express the following in terms of x and y only:

   i. $\log_{10} 24$ ii. $\log_{10} \frac{9}{8}$ iii. $\log_{10} 720$

4. (a) The streptococci bacteria population N at time t (in months) is given by $N = N_0 e^{2t}$ where N_0 is the initial population. If the initial population was 100, how long does it take for the population to reach one million?

   (b) The formula for the amount of energy E (in joules) released by an earthquake is

   $$E = 1.74 \times 10^{19} \times 10^{1.44M}$$

   where M is the magnitude of the earthquake on the Richter scale.

   i. The Newcastle earthquake in 1989 had a magnitude of 5 on the Richter scale. How many joules were released?

   ii. In an earthquake in San Francisco in the 1900's the amount of energy released was double that of the Newcastle earthquake. What was its Richter magnitude?

Worksheet 2.8 Introduction to Trigonometry

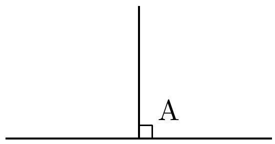
Section 1 Introduction to Plane Geometry

When two lines cross each other to make four angles all exactly the same, we call these two lines perpendicular. The angle between each of the lines is 90 deg or $\frac{\pi}{2}$ radians. We also can call an angle of $\frac{\pi}{2}$ radians a right angle, and it would be indicated on a picture by a small square in the angle.

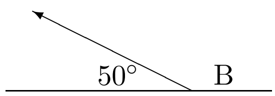
Two lines are parallel if it is possible to draw a line which is perpendicular to both lines. An arrow sitting on both lines in a diagram indicates that they are parallel.

A line forms what is called a straight angle. It is the same as if we were facing one direction, and then did an about face. We have moved through 180 deg, or π radians. To change directions on a line we must do the same thing: move through 180 deg. When we do a full turn or revolution on the plane, we move through 360 deg or 2π radians. So a straight angle is π radians and a full turn is 2π radians.

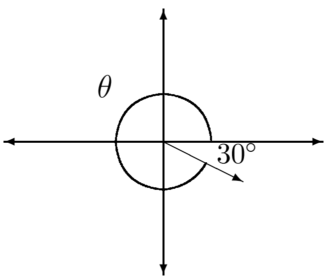
Example 1 : Angle A is a right angle.



Example 2 : Angle B is given by $B = 180° - 50° = 130°$.

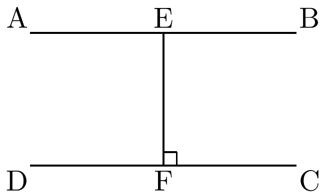


Example 3 : What is θ?



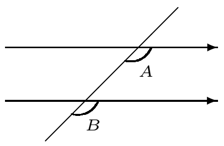
$\theta = 360° - 30°$

Example 4 : If CD and AB are parallel, and EFC is a right angle, what is FEB?

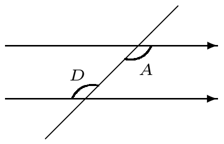


EFC is a right angle also.

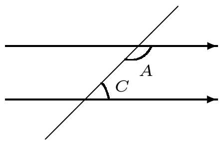
When pairs of parallel lines are both cut by another straight line, the various angles formed have some properties.



Angles A and B are called corresponding angles and are equal.

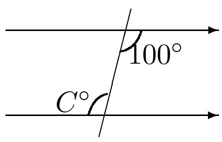


A and D are called alternate angles and are also equal.



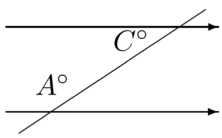
A and C are called co-interior angles and are supplementary, which means that $A + C = 180°$.

Example 5 : What is angle C?



C is 100°.

Example 6 : What is angle A, if $C° = 45°$?



$$\begin{aligned} A &= 180° - 45° \\ &= 135° \end{aligned}$$

Exercises:

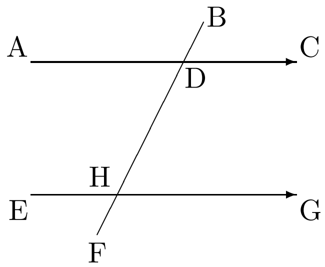
1. Name the following pairs of angles:

   (a) $\angle ADF$ and $\angle DHG$

   (b) $\angle BDC$ and $\angle DHG$

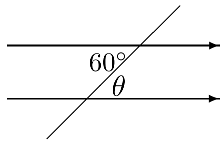
   (c) $\angle ADH$ and $\angle DHE$

   (d) $\angle BDA$ and $\angle DHE$

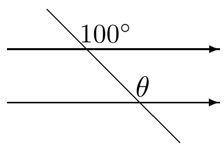


2. Find the value of θ in each of the following:

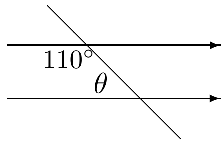
(a)



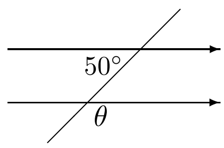
(b)



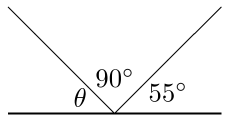
(c)



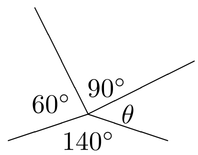
(d)



(e)



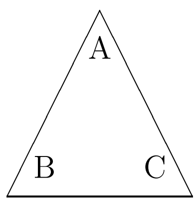
(f)



Section 2 Triangles

A triangle has three sides, which are normally denoted with lower-case letters and the opposite angle denoted by the corresponding capital letter. A triangle is often described by its three vertices. The interior angles of a triangle add up to 180 deg $= \pi$ radians. Thus, given two angles in a triangle, we can work out the third angle. If one of the angles in a triangle is a right angle, it is a right-angled triangle.

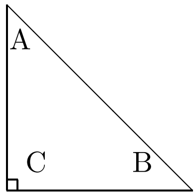
Example 1 : If angle A is 60°, and angle B is also 60°, what is angle C?



$$C = 180° - 60° - 60° = 120° - 60° = 60°$$

The triangle ABC in the above example is a special triangle called an equilateral triangle. Any triangle with three equal angles is an equilateral triangle. All the angles must be 60°. The sides, also, have an equal length.

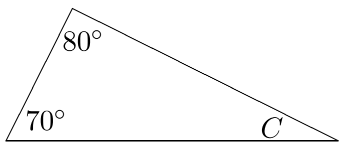
<u>Example 2</u> : If angle A is 45°, what is angle B?



Since C is a right angle, we know that $C = 90°$. Therefore $B = 180° - 90° - 45° = 45°$.

There is another special triangle called the isosceles triangle; it has two angles the same. The lengths of the sides opposite the equal angles are the same. Isosceles triangles can come in various shapes as the third angle can vary. The third angle doesn't have to be a right angle.

<u>Example 3</u> : What is the angle C?



$$C = 180° - 80° - 70° = 30°$$

You will have noticed that, when referring to angles, we have often given two units of measure: degrees and radians - the most important unit being the radian. If the unit of measure is not specified on a given angle, the angle is assumed to be in radians. To convert degrees to radians, we use the following:

$$\pi \text{ radians } = 180°$$

Therefore $1° = \frac{\pi}{180}$ radians. So

$$60° = 60 \times \frac{\pi}{180} = \frac{\pi}{3} \text{ radians}$$

A similar argument is used to convert radians to degrees.

$$\begin{aligned} \pi \text{ radians} &= 180° \\ 1 \text{ radian} &= \frac{180°}{\pi} \end{aligned}$$

Therefore, if we want to find out how many degrees there are in $\frac{\pi}{4}$ radians, we calculate as follows:

$$\frac{\pi}{4} \text{ radians} = \frac{\pi}{4} \times \frac{180°}{\pi} = 45°$$

Exercises:

1. Convert the following angles in degrees to radians; write the answers in terms of π.

   (a) $60°$
   (b) $90°$
   (c) $120°$
   (d) $45°$
   (e) $100°$
   (f) $360°$
   (g) $240°$
   (h) $80°$
   (i) $300°$

2. Convert the following angles in degrees to radians; write the answers to 2 decimal places.

   (a) $30°$
   (b) $40°$
   (c) $90°$
   (d) $100°$
   (e) $45°$
   (f) $160°$
   (g) $240°$
   (h) $600°$
   (i) $300°$

3. Convert the following angles in radians to angles in degrees:

   (a) $\frac{\pi}{2}$
   (b) $\frac{\pi}{6}$
   (c) $\frac{\pi}{8}$
   (d) 2.5
   (e) $\frac{\pi}{7}$
   (f) $\frac{\pi}{12}$
   (g) $\frac{2\pi}{3}$
   (h) $\frac{3\pi}{2}$
   (i) $\frac{5\pi}{6}$

Section 3 Pythagoras' Theorem

All right-angled triangles obey the theorem of Pythagoras, which states:

> The square of the length of the hypotenuse is equal to the sum of the squares of the other two sides.

Construct two identical squares, and divide the sides into lengths a and b as shown.

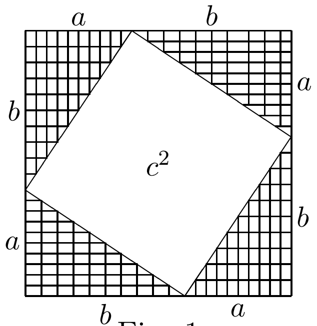


Fig. 1

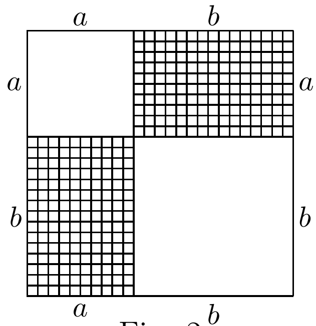
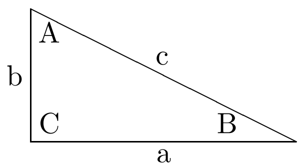


Fig. 2

$$\begin{aligned}
\text{Shaded area in Fig. 1} &= 4 \times \frac{1}{2}ab = 2ab \\
\text{Shaded area in Fig. 2} &= 2 \times ab = 2ab \\
c^2 + 2ab &= (a+b)^2 \\
&= a^2 + 2ab + b^2 \\
\text{therefore } a^2 + b^2 &= c^2
\end{aligned}$$

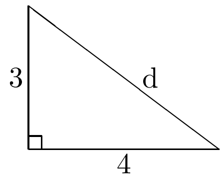
The hypotenuse is the side opposite the right angle. We now put this information onto a diagram and into a formula.



ABC is a right angled triangle, and c is the hypotenuse. Then a, b, and c satisfy the relationship:

$$a^2 + b^2 = c^2$$

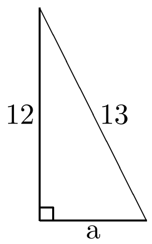
Example 4 : What is d?



$$\begin{aligned} d^2 &= 3^2 + 4^2 \\ d^2 &= 25 \\ d &= 5 \end{aligned}$$

Note that, since d is a length, we take the positive solution of the quadratic equation $d^2 = 25$.

Example 5 : What is a?

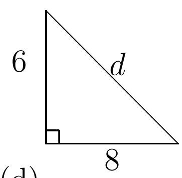


$$\begin{aligned} 13^2 &= a^2 + 12^2 \\ 169 &= a^2 + 144 \\ 169 - 144 &= a^2 \\ 25 &= a^2 \\ 5 &= a \end{aligned}$$

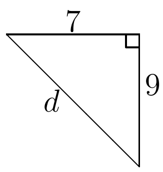
Exercises:

1. Use Pythagoras' theorem to find the length of the unknown side:

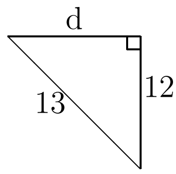
(a)



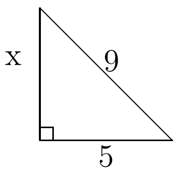
(b)



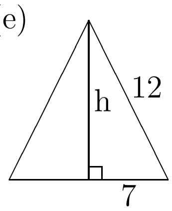
(c)



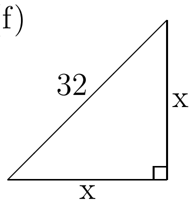
(d)



(e)



(f)

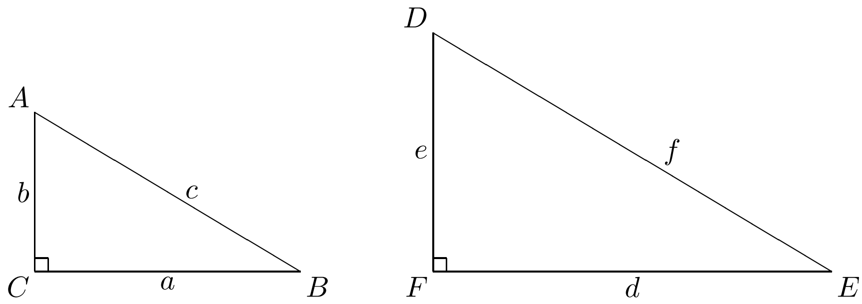


2. (a) A 4m long ladder is leaning against a wall. The foot of the ladder is 1.2 meters out from the wall. How far up the wall does the ladder reach? Draw a diagram!

   (b) In a triangle ABC, $\angle ABC = 90°$, $AC = 20$cm, and $AB = 9$ cm. Find the length of BC.

Section 4 Introductory Trigonometry

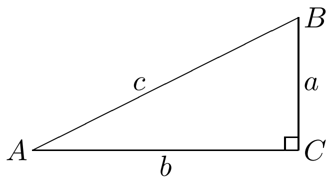
Similar triangles are ones which have the same shape. All the internal angles are the same. Similar triangles may be different sizes. The triangles ABC and DEF drawn below are similar but not the same.



The length of the sides of these triangles are proportional. That is, we can find one number p that gives

$$\begin{aligned} e &= pb \\ f &= pc \\ d &= pa \end{aligned}$$

In addition, corresponding angles are the *same*. Since the triangles formed by certain angles are proportional, we can describe angles by looking at the ratios of the sides around them. For the moment, we will restrict our discussion to that of right-angled triangles. Consider

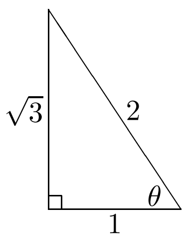


Recall that the side opposite the right angle is called the hypotenuse. In this case it is c. The side opposite the angle in question (in this case A) is called the opposite side. The remaining side is called the adjacent side, in this case b.

The trigonometric ratios are the ratios of various pairs of sides. They are sine (usually written sin), cosine (usually written cos), and tangent (usually written tan). The definitions of these ratios, in terms of right angled triangles are:

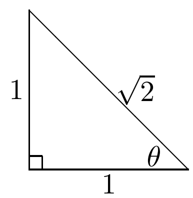
$$\begin{aligned}
\sin A &= \frac{\text{opposite side}}{\text{hypotenuse}} \\
&= \frac{a}{c} \\
\cos A &= \frac{\text{adjacent side}}{\text{hypotenuse}} \\
&= \frac{b}{c} \\
\tan A &= \frac{\text{opposite side}}{\text{adjacent side}} \\
&= \frac{a}{b}
\end{aligned}$$

Example 1 :



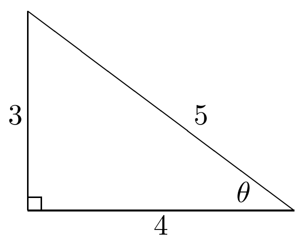
$$\begin{aligned}
\sin\theta &= \tfrac{\text{OPP}}{\text{HYP}} &= \tfrac{\sqrt{3}}{2} \\
\cos\theta &= \tfrac{\text{ADJ}}{\text{HYP}} &= \tfrac{1}{2} \\
\tan\theta &= \tfrac{\text{OPP}}{\text{ADJ}} &= \tfrac{\sqrt{3}}{1}
\end{aligned}$$

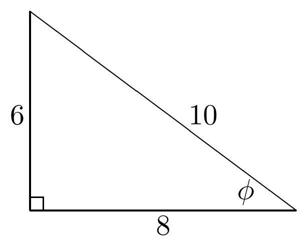
Example 2 :



$$\begin{aligned}
\sin\theta &= \tfrac{1}{\sqrt{2}} \\
\cos\theta &= \tfrac{1}{\sqrt{2}} \\
\tan\theta &= 1
\end{aligned}$$

Example 3 : How are θ and ϕ related?





Since $\sin\theta = \frac{3}{5}$ and $\sin\phi = \frac{6}{10} = \frac{3}{5}$, θ and ϕ must be the same angle.

Exercises:

1. Find the following ratios:

   (a) $\sin\theta$

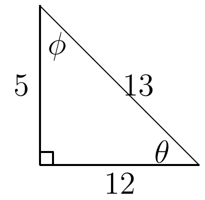
   (b) $\cos\theta$

   (c) $\tan\theta$

   (d) $\sin\phi$

   (e) $\cos\phi$

   (f) $\tan\phi$



2. Find the following ratios:

   (a) $\sin\theta$

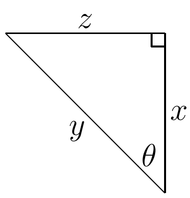
   (b) $\cos\theta$

   (c) $\tan\theta$

   (d) $\sin(90-\theta)$

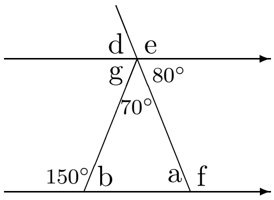
   (e) $\cos(90-\theta)$

   (f) $\tan(90-\theta)$



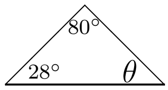
Exercises for Worksheet 2.8

1. Find the value of each of the pronumerals:

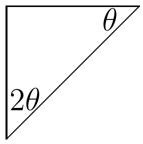


2. Find the value of θ in each of the following:

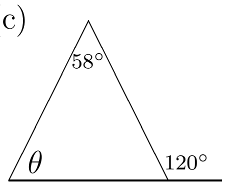
(a)



(b)



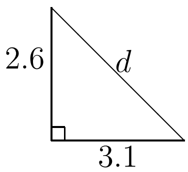
(c)



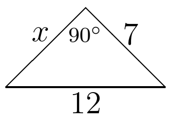
3. (a) Change the angle $\frac{7\pi}{3}$ to degrees.
   (b) Change the angle $\frac{6\pi}{5}$ to degrees.
   (c) Change 87° to radians. Write the answer to 2 decimal places.

4. Find the value of each pronumeral:

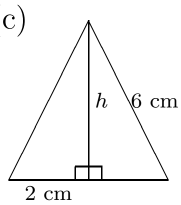
(a)



(b)

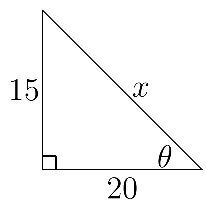


(c)



For part (c), assume the largest triangle is isosceles.

5. (a) Find the value of x
   (b) Find
      i. $\cos\theta$
      ii. $\sin\theta$
      iii. $\tan\theta$



Worksheet 2.9 Introduction to the Cartesian Plane

Section 1 The Cartesian Plane

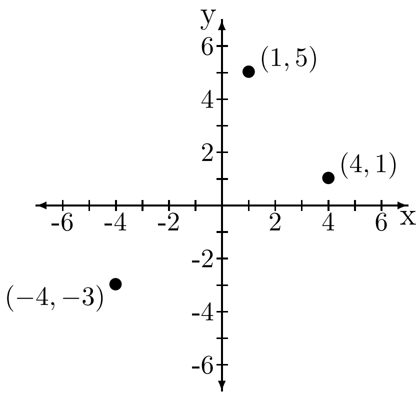
In worksheet 2.4 we discussed inequalities on the number line. This is a type of graph in one dimension. For many maths problems, we need to draw graphs in two dimensions. Graphs contain a lot of information at a glance, and so are a very useful tool.

A graph in two dimensions will arise from an equation with two variables. For example, $x + y = 0$. This equation has two variables, x and y, and there is a relationship between them which the formula expresses. To represent this equation as a graph we draw a picture of all the ordered pairs (x, y) which satisfy the relationship. The picture is placed on a cartesian plane, which is the two dimensional equivalent of the number line, and is formed by placing two number lines at right angles to each other intersecting at $(0, 0)$. This point is called the origin.

The entries in an ordered pair (x, y) are called the coordinates. In this pair, x is the first coordinate (or independent variable), and y is the second coordinate (or dependent variable). The horizontal number line represents the first coordinate of the ordered pair and the vertical line the second co-ordinate. You should label each number line (called axis) with the variable that it corresponds to.

To draw the picture, we first need to find some of the ordered pairs (x, y) which satisfy the equation. This is usually done by putting in values of x and finding the corresponding values of y. Once we have some ordered pairs, we plot them on the cartesian plane. First find the x-value, then move up or down that line to find the necessary y-value. You may well have done a similar thing when looking up a street in a street directory. Each map in a directory is a cartesian plane with letters on the horizontal axis and numbers on the vertical axis. To find $G8$ on a certain page, you find G, and then move up or down the G column until you get to the line marked 8.

Example 1 : We will plot the following ordered pairs: $(1, 5)$, $(-4, -3)$, and $(4, 1)$.



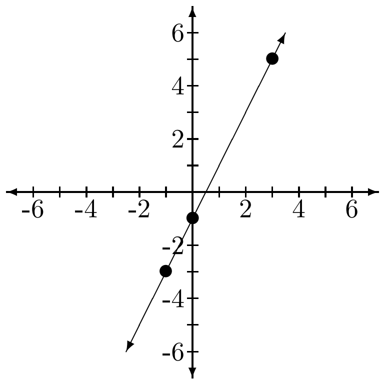
Example 2 : Plot the line $y = 2x - 1$.

We need to find some ordered pairs that satisfy the equation.

When $x = 0$, $y = -1$, so $(0, -1)$ is an ordered pair.
When $x = 3$, $y = 5$, so $(3, 5)$ is an ordered pair.
When $x = -1$, $y = -3$, so $(-1, -3)$ is an ordered pair.

We could construct a table to represent this information:

x	0	-1	3
y	-1	-3	5



Exercises:

1. Complete the table of values for the given equations.

(a) $y = x - 3$

x	0	1	2
y			

(b) $2x + y = 4$

x	0		
y		2	6

(c) $y = 5 - x$

x		2	
y	6		1

2. Complete the table of values and graph the ordered pairs on a number plane:

(a) $y = x - 3$

x	0	1	2
y			

(b) $y = x + 1$

x	0	1	2
y			

(c) $y = 6 - 2x$

x	0	1	2
y			

Section 2 Introduction to a line

The equation $ax + dy + c = 0$ represents a straight line on a cartesian plane, where a, d, and c may be any numbers. Any equation that can be put in a form that looks like this is a straight line. The important thing to notice about this equation is that the x and y variables are taken to the first power and the first power only, and they are not multiplied together; that is, there are no terms like xy, x^2, y^2, $\sqrt{x}$ or $\frac{1}{x}$.

Example 1 : Is $x + y = 0$ a straight line? Yes: we would have $a = 1$, $d = 1$, and $c = 0$.

Example 2 : Is $3x^2 + y = 2$ a straight line? No: the x variable is squared - raised to the power of 2 - so this equation is not of the required form.

Example 3 : Is $x = 5$ a straight line? Yes: we would have $a = 1$, $d = 0$, and $c = -5$.

Example 4 : Is $y = 5x$ a straight line? Yes: we would have $a = -5$, $d = 1$, and $c = 0$.

Exercises:

1. Which of the following equations represents a straight line?
   (a) $x - 2y - 6 = 0$
   (b) $xy - 8 = 0$
   (c) $\frac{1}{x} + y - 6 = 0$
   (d) $y = 2x - 1$
   (e) $x(x + y) = 4$

Section 3 Slope-Intercept Form

There is another way to write the equation of a line which allows more information about the graphed line to be readily available. We rearrange the formula $ax + dy + c = 0$ in the following way:

$$\begin{aligned} ax + dy + c &= 0 \\ ax + dy &= -c \\ dy &= -ax - c \\ y &= -\frac{a}{d}x - \frac{c}{d} \end{aligned}$$

Now since a, b and c are constants then so is $-\frac{a}{d}$ a constant together with $-\frac{c}{d}$. Let's rename these fractions by writing

$$m = -\frac{a}{d} \quad \text{and} \quad b = -\frac{c}{d}$$

Then our equation is written

$$y = mx + b$$

This form is what we will call the slope-intercept form of the straight-line equation. The slope of a line is also called the gradient, or sometimes it is referred to as rise over run. The rise of a line is the number of units up (or down) for every so many units that we move along the line. So, if a line has a slope of 2, then it rises 2 units for every unit that it goes across. If a line has a slope of $\frac{1}{2}$, then it rises 1 unit for every 2 units that it goes across. If the equation of the line is written in the form $y = mx + b$, then m is the slope, or gradient, of the line, and b is called the y-intercept. This is the value that y has when $x = 0$.

Note: Sometimes either m or b (or both!) might be equal to zero. Such an expression still represents a straight line.

What are the gradients and y-intercepts of the following straight lines?

Example 1 : $y = 2x + 3$. The gradient is 2, and the y-intercept 3.

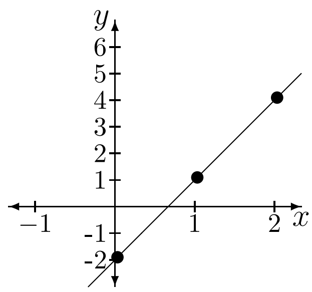
Example 2 : $y = \frac{5}{2}x + 3$. The gradient is $\frac{5}{2}$, and the y-intercept 3.

Example 3 : $y = 2 - 5x$. The gradient is -5, and the y-intercept 2.

Example 4 : $2y = 2x + 2$. The gradient is 1, and the y-intercept 1.

Let us graph the equation $y = 3x - 2$. Make a table of values for a selection of x-values, then plot the points onto a diagram, then connect them with a straight line:

x	0	1	2
y	-2	1	4

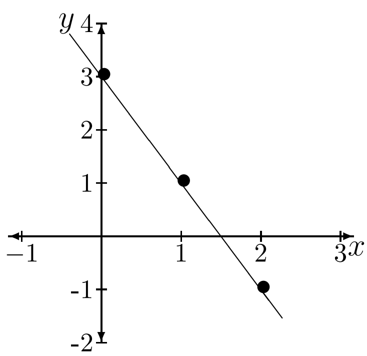


The gradient is defined as the rise over the run. Consider the rise over the run as we go from the point $(1, 1)$ to the point $(2, 4)$. The rise is $4 - 1 = 3$, and the run is $2 - 1 = 1$. Therefore the gradient is

$$\begin{aligned} \text{gradient} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

Let us graph the equation $y = -2x + 3$. Make a table of values for a selection of x-values, then plot the points onto a diagram, then connect them with a straight line:

x	0	1	2
y	3	1	-1



The gradient is defined as the rise over the run. Consider the rise over the run as we go from the point $(0, 3)$ to the point $(1, 1)$. The rise is $1 - 3 = -2$, and the run is $1 - 0 = 1$. Therefore the gradient is

$$\begin{aligned} \text{gradient} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{-2}{1} \\ &= -2 \end{aligned}$$

Two lines are parallel if they have the same gradient. If two lines are parallel, and they also have the same y-intercept, then they coincide (in other words, they are the same line). Parallel lines with different y-intercepts will never intersect.

If two lines do not have the same gradient, they will eventually meet at some point. In other words, for some x value, they will both have the same y value. This is called the point of intersection of the lines. Two lines are perpendicular if the product of the gradients of the two lines is -1 (we will not prove this). Let two lines be

$$y = m_1 x + b_1 \quad \text{and} \quad y = m_2 x + b_2$$

The two lines are perpendicular if $m_1 m_2 = -1$.

Example 5 : The lines

$$\begin{aligned} y &= 5x + 2 \\ y &= 5x + 3 \end{aligned}$$

are parallel, and never meet. They both have a gradient of 5.

Example 6 : The lines

$$\begin{aligned} y &= 5x + 4 \\ 2y &= 10x + 8 \end{aligned}$$

coincide. The second equation could be divided by 2 to become identical with the first equation.

Example 7 : The lines

$$\begin{aligned} y &= \frac{1}{5}x + 3 \\ y &= -5x + 2 \end{aligned}$$

are perpendicular, since $\frac{1}{5} \times -5 = -1$.

Example 8 : The lines

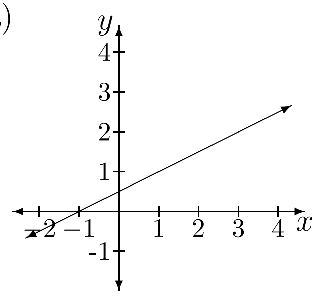
$$\begin{aligned} y &= 10x + 3 \\ y &= 5x + 10 \end{aligned}$$

meet when $x = \frac{7}{5}$ and $y = 17$. We will discuss how to find out where two lines meet in a later worksheet.

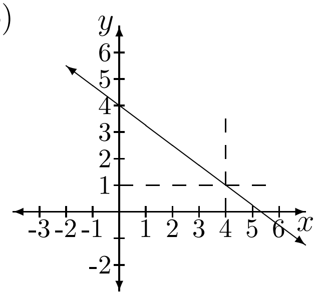
Exercises:

1. Find the gradient and y-intercept of each of the following straight lines and hence write down the equation of the straight line.

(a)



(b)



2. Write the following equations in the form $y = mx + b$, and hence find the gradient (m) and and the y-intercept (b).

(a) $y = 2x + 1$

(b) $x + y + 3 = 0$

(c) $2y = 8x - 6$

(d) $x - y + 8 = 0$

(e) $2x + 3y + 12 = 0$

(f) $\frac{1}{3}x - \frac{1}{2}y = \frac{1}{6}$

Section 4 A line through two points

Given any two points, we can find the equation of the straight line that joins them. To do so, we first work out the gradient. Recall from an earlier section that we talked about the gradient as the rise of a line divided by its run. We use this to find the gradient. The rise of a line is the difference between two y-values. The run is the difference in the corresponding x-values. The ratio of these gives the gradient. Say we are given two points on a line: (x_1, y_1) and (x_2, y_2). The slope of the line is then

$$\text{Slope } = m = \frac{y_2 - y_1}{x_2 - x_1}$$

It makes no difference which point you take as your first, but whichever one you use as your first y-value, you must use the corresponding x as your first x-value. Once you have the gradient you then use it in another rise over run ratio, except this time with just one of the points given - either point will do - and with arbitrary x and y values. So the two points we are now considering are, say, (x_2, y_2) and (x, y) and the slope m (which we now know) is given by

$$m = \frac{y - y_2}{x - x_2}$$

which we can rearrange to put it in the slope-intercept form:

$$\begin{aligned} y - y_2 &= m(x - x_2) \\ y &= mx - mx_2 + y_2 \\ y &= mx + b \end{aligned}$$

where we have let $b = -mx_2 + y_2$.

Example 1 : What is the equation of the line that $(5, 2)$ and $(4, 6)$ lie on?

$$m = \frac{2 - 6}{5 - 4} = -4$$

Now that we have found the gradient, all we have to do is find the y-intercept. Choose either of the two points that we know lie on the line. We will choose $(4, 6)$. Now substitute in values for x, y and m to find b, the y-intercept.

$$\begin{aligned} y &= mx + b \\ 6 &= (-4)4 + b \\ b &= 22 \end{aligned}$$

so the equation of the line joining $(5, 2)$ and $(4, 6)$ is $y = 22 - 4x$.

Example 2 : What is the equation of the line that $(1, 3)$ and $(2, 3)$ lie on?

$$m = \frac{3 - 3}{1 - 2} = \frac{0}{-1} = 0$$

so the gradient is 0. Now we again substitute in values of x, y and m to find b. We find that $y = 3$, so the equation of the line joining $(1, 3)$ and $(2, 3)$ is $y = (0)x + 3$ or simply $y = 3$.

Example 3 : What is the equation of the line that $(1, 5)$ and $(1, 2)$ lie on?

$$m = \frac{5 - 2}{1 - 1} = \frac{3}{0}$$

which doesn't make sense as we can't divide by zero.

If this happens, it means that the line is vertical. We can think of it as having an infinite slope. A vertical line through a point (a, b) has the equation $x = a$. So, for this example, the equation of the line is $x = 1$.

Example 4 : What is the equation of the line that $(3, 2)$ and $(-5, 2)$ lie on? The gradient is given by

$$m = \frac{2-2}{3-(-5)} = \frac{0}{8} = 0$$

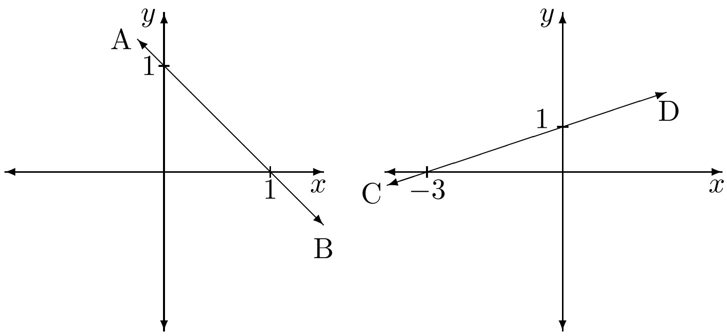
Since the slope is zero, and the line passes through a point whose y-value is 2, the equation of the line is $y = 2$.

Exercises:

1. Find the equation of the straight line passing through:
   (a) $(0, -1)$ with gradient $\frac{3}{2}$
   (b) $(2, 4)$ and $(3, 1)$
   (c) $(0, 3)$ and $(2, 7)$
   (d) $(1, 4)$ and $(3, 4)$
   (e) $(3, 1)$ and is parallel to $y = 3x - 7$
   (f) $(12, -2)$ and is perpendicular to $y = 3x - 7$

Exercises for Worksheet 2.9

1. (a) What are the gradients of the lines AB and CD in Figure 1?



   (b) Plot the points $P(-2,3)$, $Q(1,8)$, $R(-4,2)$, and $S(0,7)$. Is PQ parallel to RS? What are the slopes of PQ and RS?

   (c) Complete the table of values for the equation $y = 3x + 2$, and use the table to help you draw the graph of $y = 3x + 2$.

x	-1	0	2	
y				0

   Where does the graph intersect the x-axis?
   Where does the graph intersect the y-axis?

   (d) i. Is $y = 3x + y^2$ a straight line? Give reasons.
   ii. Is $y + 15 = 0$ a straight line? Give reasons.

   (e) What is the gradient and y-intercept of:

   i. $y = -2x - 7$ ii. $2y - 8x - 1 = 0$

   (f) What is the gradient of the line that is perpendicular to $y = 5x - 2$?

   (g) Which of the following points lies on the line $y = 2x + 12$?

   i. $(-20, 23)$ ii. $(\frac{1}{2}, 13)$

2. (a) Plot the points $X(-1, 2)$ and $Y(1, 3)$.
    i. What is the equation of the straight line that joins them?
    ii. What is the equation of the straight line which passes through X, but is perpendicular to XY?
    iii. What is the equation of the vertical line through Y?
    iv. What is the equation of the line through Y and also parallel to $y = 3x + 12$?

   (b) Find the equation of the line with gradient 2 that passes through the point $(0, -6)$.

   (c) Find the line parallel to $y = 1.5x + 4$ which goes through $(2, 4)$.

   (d) Find the line parallel to the x-axis which goes through $(22, 27)$.

   (e) What is the equation of the line that passes through the points $(-1, -15)$ and $(-10, -1)$?

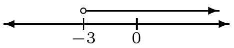
3. (a) Plot the points $A(2, 4)$, $B(1, 2)$, and $C(3, 2)$. Find the point D that would make $ABCD$ a parallelogram.

   (b) A line has gradient 1, and goes through $(5, 2)$. Does the point $(10, 7)$ lie on the line?

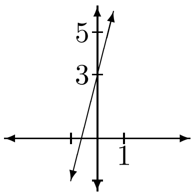
   (c) A line has gradient $-L$. Could both $(2, 5)$ and $(0, -10)$ be on the line?

Answers to Test Two
and
Exercises from Worksheets 2.1 - 2.9

Answers to Test Two

1. (a) $6uv$ (b) $3y(x+y+2)$
2. (a) $x = 12$ (b) $y = 3$
3. (a) $\frac{5x+6}{x(x+2)}$ (b) $x = -2$
4. (a)  (b) $x < -3$ or $x > 3$
5. (a) -1 (b) $\frac{2-\sqrt{2}}{2}$
6. (a) $(x+2)(x+4)$ (b) $\frac{1}{x-3}$
7. (a) Yes (b) $(x-2)(x^2+3x+7)$
8. (a) $x = 64$ (b) $3y$
9. (a) $45°$

   (b) $\sin\theta = \frac{\sqrt{3}}{2}$ $\tan\theta = \sqrt{3}$
   $\cos\theta = \frac{1}{2}$ $\theta = 60°$

10. (a) 

   (b) Slope $\frac{5}{3}$ and the y-intercept is $\frac{2}{3}$

Worksheet 2.1

Section 1

1. (a) 6 (b) $4m$ (c) u (d) $9m$ (e) $9xz$
2. (a) $12xy$ (b) $24xy$ (c) $48mnp$ (d) $48xyz$ (e) $45mn$

Section 2

1. (a) $7x + 4$
   (b) $10(2x - 1)$
   (c) $3y(6x - z)$
   (d) $6m(2n + 3p)$
   (e) $4m(4m - 1)$
   (f) $3(x^2 + 2x - 9)$
   (g) $-6(x + 4)$
   (h) $-2x(y + 4)$
   (i) $8mn(3 - 2m)$
   (j) $-xy(x + y)$
   (k) $12m^2n(1 + 2n)$
   (l) $18y^2p(4 - p)$
2. (a) $(x + 3)(4 + m)$
   (b) $(x - 1)(x + 5)$
   (c) $(y + 4)(y - 6)$
   (d) $x(x + 7)(x + 1)$
   (e) $(x - 4)(3x - 7)$

Section 3

1. (a) $\frac{x}{5}$
   (b) $\frac{x+5}{2}$
   (c) 5
   (d) x
   (e) $\frac{3x}{2}$
   (f) $\frac{x+3}{x+2}$
   (g) $\frac{a(3b+1)}{b}$
   (h) $\frac{4mn}{3}$
   (i) $\frac{p(n-2)}{3n}$
2. (a) $\frac{3(x+3)}{2}$
   (b) $\frac{3(x-5)}{8}$
   (c) $\frac{3p^2}{2}$
   (d) $\frac{4(3m+4)}{3(p+2)}$
   (e) $\frac{4}{3}$
   (f) $2x$
   (g) $\frac{3q}{8}$
   (h) $\frac{5(x+y)}{24y}$

Exercises 2.1

1. 1, 2, 3, 6, 9, 18
2. (a) 1, 2, 4, 8
   (b) 1, 2, t, $2t$
3. (a) 4 (b) 8 (c) 1

4. (a) $-3(x-7)$
   (b) $3x(2x+1)$
   (c) $6x(3x+2y)$
   (d) $6m(t-4m)$
   (e) $9(2x+3y)$
   (f) $(x+3)(x-7)$

5. (a) $x+4$
   (b) $\frac{x(y+3)}{2}$
   (c) $\frac{7n}{3}$
   (d) 2
   (e) $\frac{819x^2}{16y}$
   (f) $3pq$

Worksheet 2.2

Section 1

1. (a) 11
   (b) 18
   (c) 0.8
   (d) $2\frac{1}{4}$
   (e) -6
   (f) -10
   (g) -1.3
   (h) 1.4
   (i) $\frac{1}{2}$
   (j) 40

Section 2

1. (a) 5
   (b) 4
   (c) 15
   (d) 6
   (e) $5\frac{1}{2}$
   (f) -8
   (g) -14
   (h) -16
   (i) 25
   (j) -8

Section 3

1. (a) 5
   (b) 24
   (c) 5
   (d) $-2\frac{1}{2}$
   (e) -8
   (f) 22
   (g) $1\frac{1}{3}$
   (h) -14
   (i) 19
   (j) $18\frac{3}{4}$

Exercises 2.2

1. (a) $x=-11$
   (b) $x=-11$
   (c) $y=3$
   (d) $t=18$
   (e) $y=\frac{41}{6}$
   (f) $x=-5$
   (g) $x=9$
   (h) $x=28$
   (i) $x=\frac{5}{2}$
   (j) $y=\frac{-7}{2}$
   (k) $x=\frac{3}{5}$
   (l) $x=3\frac{2}{7}$
   (m) $m=6\frac{2}{3}$
   (n) $y=-17$
   (o) $y=\frac{2}{5}$
   (p) $x=22$
   (q) $x=\frac{1}{2}$
   (r) $t=\frac{3}{4}$
   (s) $t=-32$
   (t) $T=18$

2. (a) -1
   (b) 46,47
   (c) 22,24
   (d) \$5.33
   (e) 40
   (f) 6

Worksheet 2.3

Section 1

1. (a) $\frac{5x}{6}$
   (b) $\frac{-2m}{35}$
   (c) $\frac{13t}{10}$
   (d) $\frac{m+10}{12}$
   (e) $\frac{13m+1}{14}$
   (f) $\frac{2y}{(y+1)(y+3)}$
   (g) $\frac{9t-11}{(t+1)(t-3)}$
   (h) $\frac{m(7m+31)}{(m+4)(m+5)}$
   (i) $\frac{-y+3)}{(y+1)(y+2)}$
   (j) $\frac{35y+8}{20xy}$

Section 2

1. (a) $\frac{3}{20}$
   (b) $\frac{1}{2}$
   (c) $4\frac{1}{2}$
   (d) $\frac{9y}{14}$
   (e) $\frac{7q}{10}$
   (f) $1\frac{1}{5}$
   (g) $\frac{10}{21y}$
   (h) $\frac{5(m+5)}{6(m+1)}$
   (i) $\frac{12}{q(p+1)}$
   (j) $\frac{8(x+1)}{3}$

Section 3

1. (a) $x = -13$ (b) $x = 8$ (c) $x = 5\frac{5}{7}$ (d) $x = 1$ (e) $x = -1\frac{1}{2}$
2. (a) $x = \frac{8}{3y}$ (b) $x = \frac{4y+5}{3}$ (c) $x = \frac{4y-19}{3}$ (d) $x = \frac{1-5y}{3y}$ (e) $x = \frac{7}{y(5+3y)}$

Exercises 2.3

1. (a) $\frac{7x}{12}$
   (b) $\frac{y^2+4}{xy^3}$
   (c) $\frac{-9(x+1)}{2}$
   (d) $\frac{-(b+2)}{(b-1)(b-2)}$
   (e) $2x$
   (f) $\frac{2(1+2x)}{x(x-4)}$
   (g) $\frac{1}{x+1}$
   (h) $\frac{x^2+4x+5}{(x+1)(x+2)(x+3)}$
   (i) $\frac{37a+28}{28}$
   (j) $\frac{-5p}{6}$

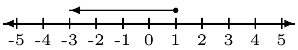
2. (a) $\frac{-7x+38}{6}$
   (b) $\frac{2(x+3)}{5}$
   (c) $\frac{24(x-3)}{x+7}$
   (d) $\frac{3y}{2}$
   (e) $\frac{3(5m-7)}{4(m+2)}$
   (f) $-\frac{9}{2}$

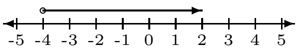
Worksheet 2.4

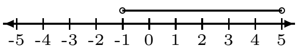
Section 1

1. (a) $x \leq 6$ (b) $-1 \leq x \leq 4$ (c) $2 < x < 5$

2. (a) $3, 3\frac{1}{2}, 4\frac{1}{2}$ (b) $-7, -8\frac{1}{2}$ (c) $-4, 0, \frac{1}{2}$ (d) $2\frac{1}{4}, 4, 5\frac{1}{4}, 5.5$

3. (a) 

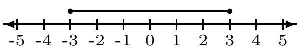
   (b) 

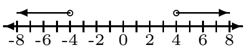
   (c) 

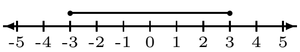
Section 2

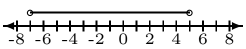
1. (a) $x < 12$ (b) $x > 2\frac{1}{2}$ (c) $m \leq -\frac{7}{3}$ (d) $m \geq 7$ (e) $m < \frac{2}{3}$ (f) $y \geq -4$ (g) $y < 19$ (h) $x \leq 3$ (i) $x > -\frac{113}{41}$ (j) $m > -\frac{1}{7}$

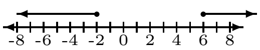
Section 3

1. (a) 

   $-3 \leq x \leq 3$
   (b) 

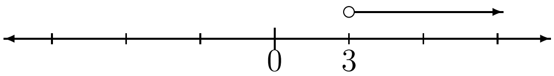
   $x < -4$ or $x > 4$
   (c) 

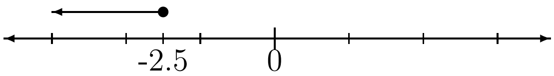
   $-3 \leq x \leq 3$
   (d) 

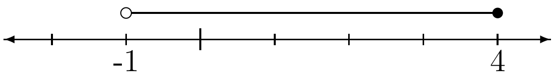
   $-7 < x < 5$
   (e) 

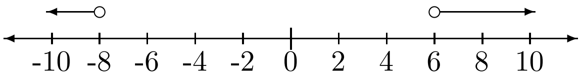
   $x \leq 2$ or $x \geq 6$

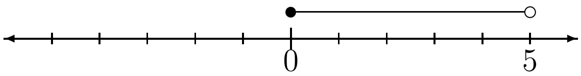
Exercises 2.4

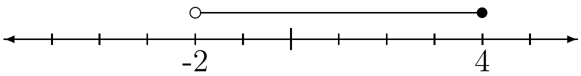
1. (a) i. 

ii. 

iii. 

iv. 

v. 

vi. 

(b) i. $x > 1$ or $x \leq -0.5$
ii. $-1 < x \leq 3$
iii. $S \geq 0.07$
iv. $T \leq 2$
v. $D \leq 200$
vi. $x \geq 18$ or $x < 1$
vii. $A \geq 35$

(c) (i) No (ii) No

2. (a) $x \leq 5$
(b) $y < -6$
(c) $x < 12$
(d) $-3 < x < 3$
(e) $a < 15$
(f) $x < -\frac{3}{2}$
(g) $t \leq -13$
(h) $x \leq 12$
(i) $x = 7$ or $x = -7$
(j) $x = -1$ or $x = 13$
(k) $y > 2$ or $y < -2$
(l) $-2 < x < 1$
(m) $x \geq 10$ or $x \leq 0$
(n) No solutions

Worksheet 2.5

Section 1

1. (a) $2\sqrt{3}$ (b) $5\sqrt{5}$ (c) $4\sqrt{3}$ (d) $6\sqrt{2}$ (e) $3\sqrt{3}$

2. (a) $3\sqrt{2}+\sqrt{10}$
   (b) $6\sqrt{3}$
   (c) $4\sqrt{5}+12$
   (d) $5+3\sqrt{3}$
   (e) $19-9\sqrt{5}$
   (f) 23
   (g) $4+2\sqrt{3}+2\sqrt{5}+\sqrt{15}$
   (h) $1+\sqrt{3}-\sqrt{2}-\sqrt{6}$
   (i) 62
   (j) $8+2\sqrt{15}$

Section 2

1. (a) $\frac{9\sqrt{2}+4}{20}$
   (b) $\frac{12\sqrt{3}+39}{35}$
   (c) $\frac{3\sqrt{2}}{2}$
   (d) $\frac{\sqrt{5}-28}{20\sqrt{2}}$
   (e) $\frac{7\sqrt{2}+23}{(\sqrt{2}+1)(\sqrt{2}+5)}$
   (f) $\frac{9\sqrt{3}+13\sqrt{7}}{(\sqrt{3}+\sqrt{7})(\sqrt{3}+2\sqrt{7})}$
   (g) $\frac{7\sqrt{5}+16}{6}$
   (h) $\frac{-7\sqrt{5}-25}{63}$
   (i) $\frac{-16+8\sqrt{2}+\sqrt{6}-\sqrt{3}}{(\sqrt{2}-1)(\sqrt{2}-3)}$
   (j) $\frac{6\sqrt{2}-42-5\sqrt{10}-5\sqrt{5}}{2\sqrt{5}}$

Section 3

1. (a) $\frac{3\sqrt{5}}{5}$
   (b) $\sqrt{2}$
   (c) $\frac{3\sqrt{3}}{4}$
   (d) $\frac{2+\sqrt{2}}{2}$
   (e) $\frac{\sqrt{5}(\sqrt{3}-1)}{5}$
   (f) $-\frac{2\sqrt{2}}{3}$
   (g) $\frac{\sqrt{10}(\sqrt{5}+3)}{10}$
   (h) $\frac{\sqrt{7}(\sqrt{2}-1)}{7}$
   (i) $\frac{\sqrt{3}+1}{2}$
   (j) $2(\sqrt{6}+2)$
   (k) $\frac{7(\sqrt{7}+2)}{3}$
   (l) $-\frac{3(\sqrt{5}-1)}{4}$
   (m) $\frac{\sqrt{5}(\sqrt{2}+3)}{5}$
   (n) $-\frac{(1+\sqrt{5})}{2}$
   (o) $-\frac{(\sqrt{3}+2)(\sqrt{3}-4)}{13}$
   (p) $\frac{(5+2\sqrt{3})(\sqrt{5}-3)}{2}$

Exercises 2.5

1. (a) 2
   (b) 5
   (c) $\sqrt{6}$
   (d) $8\sqrt{3}$
   (e) $6-2\sqrt{7}$
   (f) $-1-\sqrt{2}$
   (g) $\sqrt{3}-4$
   (h) $2\sqrt{2}$
   (i) $|a|$
   (j) $7b^2$
   (k) $\frac{p}{4}$
   (l) $\frac{25}{2}$

2. (a) $13\sqrt{3}$
(b) $-12\sqrt{5}$
(c) $3\sqrt{5}$
(d) 20
(e) 1
(f) $30+12\sqrt{6}$
(g) $9+7\sqrt{5}$
(h) $5\sqrt{15}+1$
(i) $\frac{2\sqrt{3}}{3}$
(j) $\frac{\sqrt{5}-1}{2}$
(k) $2+\sqrt{3}$
(l) $\frac{10\sqrt{2}-3}{12}$
(m) $-3(\sqrt{3}+3)$
(n) $-\frac{2}{7}(11\sqrt{2}-9)$

3. (a) $19+8\sqrt{3}$
(b) $34+9\sqrt{2}$
(c) 0.71
(d) 0.81
(e) 87.96 sq cm
(f) 1.83

Worksheet 2.6

Section 1

1. (a) $6(x+4)$
(b) $4x(2x-1)$
(c) $2xy(3+5x)$
(d) $m^2(m^2-3)$
(e) $2x(3x+4+6y)$
(f) $(4m+5)(2m-3)$
(g) $(x+5)(x+2)$
(h) $(m-4)(m+3)$
(i) $(t-2)(2t+1)$
(j) $(2y-5)(3y+2)$

Section 2

1. (a) x^2-4
(b) y^2-25
(c) y^2-36
(d) x^2-49
(e) $4x^2-1$
(f) $9m^2-16$
(g) $9y^2-25$
(h) $4t^2-49$

2. (a) $(x+4)(x-4)$
(b) $(y+7)(y-7)$
(c) $(x+5)(x-5)$
(d) $(2x+5)(2x-5)$
(e) $(4+y)(4-y)$
(f) $(m+6)(m-6)$
(g) $(2m+7)(2m-7)$
(h) $(3m+4)(3m-4)$

3. (a) $x^2+10x+25$
(b) $x^2+18x+81$
(c) y^2-4y+4
(d) m^2-6m+9
(e) $4m^2+20m+25$
(f) $t^2+20t+100$
(g) $y^2+16y+64$
(h) $t^2+12t+36$

4. (a) $(y-3)^2$
(b) $(x-5)^2$
(c) $(x+4)^2$
(d) $(x+10)^2$
(e) $(m+8)^2$
(f) $(t-15)^2$
(g) $(m-6)^2$
(h) $(t+9)^2$

Section 3

1. (a), (b), (e), and (f)

Section 4 part 1

1. (a) $(x+3)(x+1)$
   (b) $(x+11)(x+4)$
   (c) $(x+13)(x-2)$
   (d) $(x+10)(x-3)$
   (e) $(x+6)(x+4)$
   (f) $(x-12)(x-2)$
   (g) $(x-5)(x-2)$
   (h) $(x-8)(x+3)$
   (i) $(x+5)(x-3)$
   (j) $(x-5)(x+3)$

Section 4 part 2

1. (a) $(2x+3)(x+4)$
   (b) $(3x+1)(x+5)$
   (c) $(2x+3)(3x+4)$
   (d) $(2x+5)(x+2)$
   (e) $(3x+2)(4x+1)$
   (f) $(2x+1)(x-3)$
   (g) $(3x+2)(x-4)$
   (h) $(3x+4)(x-5)$
   (i) $(5x+2)(x+3)$
   (j) $(5x+2)(2x+3)$

Section 5

1. (a) $3(x-\frac{-2+\sqrt{52}}{6})(x-\frac{-2-\sqrt{52}}{6})$
   (b) $(x-\frac{-3+\sqrt{5}}{2})(x-\frac{-3-\sqrt{5}}{2})$
   (c) $2(x-\frac{-8+\sqrt{40}}{4})(x-\frac{-8-\sqrt{40}}{4})$
   (d) $3(x-\frac{-5+\sqrt{13}}{6})(x-\frac{-5-\sqrt{13}}{6})$
   (e) $3(x-\frac{-6+\sqrt{12}}{6})(x-\frac{-6-\sqrt{12}}{6})$
   (f) $5(x-\frac{-7+\sqrt{89}}{10})(x-\frac{-7-\sqrt{89}}{10})$
   (g) $3(x-\frac{-5+\sqrt{73}}{6})(x-\frac{-5-\sqrt{73}}{6})$
   (h) $2(x-\frac{-4+\sqrt{8}}{4})(x-\frac{-4-\sqrt{8}}{4})$
   (i) $5(x-\frac{-2+\sqrt{44}}{10})(x-\frac{-2-\sqrt{44}}{10})$
   (j) $2(x-\frac{-1+\sqrt{57}}{4})(x-\frac{-1-\sqrt{57}}{4})$

Section 6

1. (a) x
   (b) $\frac{3x^2-4}{x}$
   (c) $\frac{x+1}{3}$
   (d) $\frac{x+2}{x+3}$
   (e) $\frac{x-4}{2}$
   (f) $\frac{x-3}{6}$
   (g) $\frac{x+5}{x+2}$
   (h) $\frac{2(x-4)}{x+2}$
   (i) $\frac{x^2}{3}$
   (j) $\frac{2x+3}{x+3}$

2. (a) $\frac{5x^2+13x+9}{(x+2)(x+3)}$ (b) $\frac{2(2x^2+3x+5)}{(x+2)(x-5)}$ (c) $\frac{2(x^2+5x+5)}{(x+2)(x+4)}$ (d) $\frac{2(4x+17)}{(x+2)(x+3)(x+5)}$ (e) $\frac{3x+17}{(x+2)(x+3)(x-5)}$ (f) $\frac{-1}{x+3}$ (g) 0 (h) $\frac{-x^2-6x+9}{2(x-3)}$ (i) $\frac{-x(x-1)}{(x+1)(x+3)}$ (j) $\frac{-2(x-1)}{(x+6)}$

3. (a) 4, 2 (b) -5, -3 (c) -4, -3 (d) -11, 2 (e) 4, 3 (f) -2, $\frac{3}{2}$ (g) 7, $-\frac{1}{2}$ (h) 4, $-\frac{2}{3}$ (i) -2, $\frac{1}{7}$ (j) 11, 7

4. (a) $\frac{3\pm\sqrt{5}}{2}$ (b) $\frac{3}{2} \pm \frac{\sqrt{92}}{4}$ (c) $-\frac{1}{3} \pm \frac{\sqrt{28}}{6}$ (d) $\frac{13}{4} \pm \frac{\sqrt{113}}{4}$

Section 7

1. (a) $\frac{(x+4)(x-3)}{x+2}$ (b) $\frac{4}{3(x-4)}$ (c) $\frac{x(2x+3)}{5(x+5)}$ (d) $\frac{x(3x+2)}{5(2x+1)}$ (e) $\frac{(x-4)(x+2)}{(2x-1)(x+7)}$

Exercises 2.6

1. (a) $x^2 - 4$ (b) $4x^2 - 16y^2$

2. (a) $(12x - y)(12x + y)$
   (b) $(4a - 3b)(4a + 3b)$
   (c) $(x + 5)(x + 2)$
   (d) $(b + 7)(b + 2)$
   (e) $(x + 7)^2$
   (f) $(x + 3)(x - 2)$
   (g) $2x(x - 3)(x + 8)$
   (h) $(x + 3)(2x + 1)$
   (i) $(b + 3)(5b + 2)$
   (j) $2(3y + 1)^2$
   (k) $6(x + 1)(2x - 1)$
   (l) $(7a + 5)(a - 2)$
   (m) $(x + 2)(1 - x)$
   (n) $(2x + 1)(2 - x)$

3. (a) $[x - (\frac{-5-\sqrt{13}}{2})][x - (\frac{-5+\sqrt{13}}{2})]$ or $(\frac{2x+5+\sqrt{13}}{2})(\frac{2x+5-\sqrt{13}}{2})$ or $\frac{1}{4}(2x + 5 + \sqrt{13})(2x + 5 - \sqrt{13})$
   (b) $(2x + 1)(x - 1)$
   (c) $\frac{A+2}{3}$
   (d) $\frac{2(x+2)}{2x+3}$
   (e) $\frac{x(x+2)}{x+3}$
   (f) $\frac{x(x+3)}{(2x+1)(x+1)}$
   (g) $\frac{4x^2-3x+9}{(x+3)^2(x-3)}$
   (h) $x = -3$ or $x = 1$

Worksheet 2.7

Section 1

1. (a) $x = 3^9$ (b) $8 = 2^x$ (c) $27 = 3^x$ (d) $x = 4^3$ (e) $y = 2^5$ (f) $y = 5^2$

2. (a) $4 = \log_3 y$ (b) $x = \log_3 27$ (c) $2 = \log_4 m$ (d) $5 = \log_3 y$ (e) $5 = \log_x 32$ (f) $x = \log_4 64$

3. (a) 81 (b) 3 (c) 10 (d) 64 (e) 243 (f) 8

Section 2

1. (a) $\log_2 \frac{y}{x}$ (b) $4 + 3\log_2 x$ (c) $\log_3 y - 1$ (d) $2\log_4(xy)$ (e) 0

2. (a) 8 (b) $\frac{6}{5}$ (c) $\frac{1}{8}$ (d) $\frac{2}{11}$ (e) $1\frac{1}{2}$

Section 3

1. (a) 0.34 (b) -0.14 (c) 0.89 (d) 1.86 (e) 44.70 (f) 1.62 (g) 9.03 (h) 0.10 (i) 3.41 (j) 4.65

2. (a) $\ln xy^3$ (b) $\frac{1}{2}\ln 8 + 2\ln x$ (c) 1 (d) $2e^5$ (e) 2 (f) $2 + \ln 3$

3. (a) 14.88 (b) 5.42 (c) 9.03 (d) 1.84 (e) -0.71

Exercises 2.7

1. (a) 3 (b) 0 (c) 3 (d) -2 (e) x
2. (a) 16 (b) $\frac{1}{81}$ (c) 49.5 or $\frac{99}{2}$ (d) 6 (e) 3
3. (a) 21
   (b) 2
   (c) i. $x+y$ ii. $2y-x$ iii. $2y+x+1$
4. (a) 4.6054 months
   (b) i. 2.76×10^{26} Joules ii. 5.2 on the Richter scale.

Worksheet 2.8

Section 1

1. (a) alternate (b) corresponding (c) co-interior (d) corresponding
2. (a) $60°$ (b) $100°$ (c) $70°$ (d) $130°$ (e) $35°$ (f) $70°$

Section 2

1. (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$ (e) $\frac{5\pi}{9}$ (f) 2π (g) $\frac{4\pi}{3}$ (h) $\frac{4\pi}{9}$ (i) $\frac{5\pi}{3}$
2. (a) 0.52 (b) 0.70 (c) 1.57 (d) 1.75 (e) 0.79 (f) 2.79 (g) 4.19 (h) 10.47 (i) 5.24
3. (a) $90°$ (b) $30°$ (c) $22.5°$ (d) $143°$ (e) $26°$ (f) $15°$ (g) $120°$ (h) $270°$ (i) $150°$

Section 3

1. (a) 10
   (b) 11.40
   (c) 5
   (d) 7.48
   (e) 9.75
   (f) 22.63

2. (a) 3.816 metres
   (b) 17.86 cm

Section 4

1. (a) $\frac{5}{13}$
   (b) $\frac{12}{13}$
   (c) $\frac{5}{12}$
   (d) $\frac{12}{13}$
   (e) $\frac{5}{13}$
   (f) $\frac{12}{5}$

2. (a) $\frac{z}{y}$
   (b) $\frac{x}{y}$
   (c) $\frac{z}{x}$
   (d) $\frac{x}{y}$
   (e) $\frac{z}{y}$
   (f) $\frac{x}{z}$

Exercises 2.8

1. $a = 80°$
   $b = 30°$
   $d = 80°$
   $e = 100°$
   $f = 100°$
   $g = 30°$

2. (a) 72° (b) 30° (c) 62°

3. (a) 420° (b) 216° (c) 1.52

4. (a) 4.05 (b) 9.75 (c) 5.66

5. (a) 25
   (b) i. $\frac{4}{5}$ ii. $\frac{3}{5}$ iii. $\frac{3}{4}$

Worksheet 2.9

Section 1

1. (a)

x	0	1	2
y	−3	−2	−1

(b)

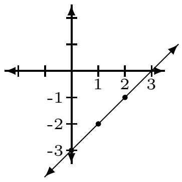
x	0	1	−1
y	4	2	6

(c)

x	−1	2	4
y	6	3	1

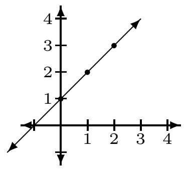
2. (a)

x	0	1	2
y	-3	-2	-1



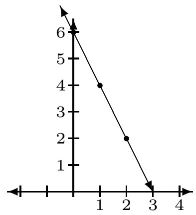
(b)

x	0	1	2
y	1	2	3



(c)

x	0	1	2
y	6	4	2



Section 2

1. (a) & (d)

Section 3

1. (a) $m = \frac{1}{2}$, $b = \frac{1}{2}$
   $y = \frac{1}{2}x + \frac{1}{2}$

   (b) $m = -\frac{3}{4}$, $b = 4$
   $y = -\frac{3}{4}x + 4$

2. (a) $m = 2$, $b = 1$
   (b) $m = -1$, $b = -3$
   (c) $m = 4$, $b = 3$
   (d) $m = 1$, $b = 8$
   (e) $m = -\frac{2}{3}$, $b = -4$
   (f) $m = \frac{2}{3}$, $b = -\frac{1}{3}$

Section 4

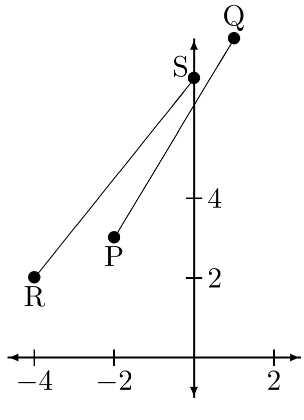
1. (a) $y = \frac{3}{2}x - 1$
   (b) $y = -3x + 10$
   (c) $y = 2x + 3$
   (d) $y = 4$
   (e) $y = 3x - 8$
   (f) $y = -\frac{1}{3}x + 2$

Exercises 2.9

1. (a) i. Line AB has slope -1
   ii. Line CD has slope $\frac{1}{3}$

   (b) Line PQ has slope $1\frac{2}{3}$
   Line RS has slope $1\frac{1}{4}$

PQ and RS are not parallel



(c)

x	-1	0	2	$-2/3$
y	-1	2	8	0

x-intercept $-\frac{2}{3}$ y-intercept 2

(d) i. No ii. Yes

(e) i. $m = -2$, y intercept -7 ii. $m = 4$, y intercept $\frac{1}{2}$

(f) $-\frac{1}{5}$

(g) i. No ii. Yes

2. (a) i. $y = 0.5x + 2.5$
   ii. $y = -2x$
   iii. $x = 1$
   iv. $y = 3x$

   (b) $y = 2x - 6$

   (c) $y = 1.5x + 1$

   (d) $y = 27$

   (e) $y = \frac{-14}{9}x - \frac{149}{9}$

3. (a) $D(4, 4)$ There are other choices.

   (b) $(10, 7)$ is on the line

   (c) Only for $L = -\frac{15}{2}$