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**Title**: Global Extension of Local Holomorphic Isometries with respect to the Bergman Metric


**Abstract**:

By a celebrated work of Calabi's every germ of holomorphic isometry of a simply-connected Kähler manifold into the projective space $\mathbb{P}^n$ extends to a global isometry. The same remains true when the target manifold is replaced by an irreducible compact Hermitian symmetric manifold, since the latter can be holomorphically isometrically embedded into some $\mathbb{P}^n$.

By a rigidity result of the same work of Calabi's, any local holomorphic isometry of a Hermitian symmetric manifold $X$ of the compact type into the projective space $\mathbb{P}^n$ is equivariant with respect to the isometry group of $X$. For instance, any local holomorphic isometry between projective spaces equipped with Fubini-Study metrics must be congruent to a Veronese embedding. For the dual situation of germs of holomorphic isometeric immersions between bounded symmetric domains it is commonly believed that the situation is even more rigid. For instance, using the notion of the *diastasis* introduced by Calabi in the afore-mentioned work Umehara proved that any local holomorphic isometry between complex unit balls equipped with the Bergman metric must be necessarily totally geodesic. However, since a higher-rank bounded symmetric domain *cannot* be holomorphically isometrically immersed into the complex unit ball because of the monotonicity property of holomorphic bisectional curvatures, the general problem remained unresolved.

More generally, let $f : D \to D'$ be a germ of holomorphic isometry up to a normalizing constant between two bounded domains equipped with the Bergman metric. We pose the question of characterizing such maps and of finding conditions which force the map to be totally geodesic. The special case of the problem where $D$ is the unit disk, $D'$ is a polydisk, and $f$ satisfies some supplementary conditions, was studied by Clozel-Ullmo in connection to a problem in arithmetic geometry. There first of all they proved that $f$ extends to an algebraic map by making use of real algebraic functional identities arising from isometries up to scalar constants.

We have now developed a general method for the analytic continuation of germs of holomorphic isometries. Starting with the same functional identities and polarizing we obtain in the general situation an infinite number of holomorphic identities, and the first question is to determine whether the functional identities are sufficiently nondengenerate to force analytic continuation. We solve this problem by studying deformations of solutions of the holomorphic functional identities, and force analytic continuation by showing that, in the event that there are nontrivial deformations, the germ of holomorphic isometry must take values on linear sections of the embedding of the domain into the infinite-dimensional projective space $\mathbb{P}^\infty$. This statement applies to all germs of holomorphic isometries (up to scalar constants) with respect to the Bergman metric.

The linear sections that we obtain correspond to extremal functions with respect to the Bergman metric. In the event that the Bergman kernel function $K(z, \overline{w})$ is rational in $(z, \overline{w})$, they will yield algebraic equations satisfied by the germ of map. This is the case for bounded symmmetric domains, which implies in this case that the germ of holomorphic map admits a global extension as a proper algebraic map. We study the asymptotic behavior of such maps and show that in certain cases such maps are necessarily totally geodesic.

In contrast to the dual case of Hermitian symmetric manifolds of the compact type, between certain bounded symmetric domains we have produced examples of holomorphic isometric embeddings which are *not* totally geodesic. The simplest examples, which disprove a Conjecture of Clozel-Ullmo's, are holomorphic isometric embeddings of the unit disk into polydisks whose algebraic extensions develop branch points on the unit circle. On the other hand, again in constrast to the dual case, any equivariant holomorphic isometric embeddings between two bounded symmetric domains must be totally geodesic.