Speaker: Pascal Auscher (Université de Paris-Sud)

Title: On $T(b)$ Theorems

Abstract:

A $T(b)$ theorem is a generic expression to designate a criterion for L^2 boundedness of singular integral operators. Such operators arise in many contexts: complex analysis (the Hilbert transform, the Cauchy transform, the Beurling transform, ...), classical partial differential equations, geometry (Riesz transforms on $\mathbf{R}^n$, on Lie groups or on manifolds) and Fourier multipliers. They usually have a kernel representation but one also encounters singular operators without good kernel representation (Kato conjecture). L^2 boundedness is usually the crucial fact as it has many consequences (L^p inequalities, maximal inequalities, a.e. convergence, ...).

Establishing L^2 boundedness is either a trivial fact, for example by Fourier methods (Hilbert, Beurling transforms, Fourier multipliers), or a very difficult task (Cauchy transform, Kato operator). The remarkable $T(b)$ theorem due to David, Journé and Semmes in 1985 after pioneering work by Meyer and McIntosh, says that the boundedness is equivalent to a mild condition called weak boundedness property and a controlled action on **one** non degenerate bounded test function b (for example, $b = 1$ but many more examples are possible), namely that $T(b)$ belongs to the space BMO, and the same for the adjoint T^* with possibly a different b. The non degeneracy is that this function b should have a controlled oscillation at all scales and all locations. Such statements were motivated by the recent proof of the L^2 boundedness of the Cauchy integral operator on Lipschitz curves by Coifman, McIntosh and Meyer in 1982. This was used by David in 1983 to prove a conjecture of Calderón: in other words, he could check there is such a b with the desired properties.

But the story continued. The next impetus was given by Christ in 1991 when he observed that such a test function b could be replaced by a family (b_Q) indexed by cubes Q of non degenerate bounded functions with bounded image under the operator. This theorem was proved valid in the rather general setting of a metric measure space with doubling measure. The application he had in mind was the Painlevé conjecture for the analytic capacity that was solved later on by Tolsa in 2003 with Christ's $T(b)$ strategy. The difficulty is that the complex plane is equipped with a measure with linear growth but not necessarily doubling. Still Tolsa and Nazarov, Treil, and Volberg had derived appropriate extensions of Christ's result to those settings.

At the same time, the Kato conjecture was still an open problem. It has to do with the boundedness and invertibility of the Dirichlet to Neumann map at the boundary for some elliptic PDE's with non-smooth bounded coefficients on domains of $\mathbf{R}^n$. It involves establishing a certain inequality for the square root of some elliptic operators. In one dimension, this problem is tied to the Cauchy integral and in fact it can be shown that the David-Journé-Semmes $T(b)$ theorem applies. Nevertheless, in higher dimensions something else had to be done because of the lack of kernel representation. Still there is a possible reduction to a $T(b)$ theorem for square roots formulated by Tchamitchian and myself around 1997 in the spirit of Christ's but with a control in L^2 rather than L^∞. Then Hofmann came up with the essential idea to exhibit the needed test functions b_Q. Gathering the efforts of the three of us with those of Lacey and McIntosh, the solution was given in 2001. This strategy has been succesfully continued by McIntosh and collaborators for other boundary value problems and functional calculi of perturbed Dirac operators. The solution also opened the door to a further generalization of Christ's statement for singular integral operators: the L^∞ condition on b_Q and Tb_Q can be replaced by an $L^2(Q)$ condition as proved by Hofmann, Muscalu, Thiele, Tao and myself. But we placed ourselves in a model situation of perfect dyadic cancellation, believing (may be wrongly) that the general case was a straightforward extension. A recent work by a Chinese mathematician Q. Yang and myself shows however that there is a simple reduction to the model case via the Beylkin-Coifman-Rokhlin algorithm used in applications of singular integrals to numerical analysis. This new $T(b)$ theorem permits us to establish the L^2 boundedness of layer potentials arising from a class of elliptic equations as proved recently by Alfonseca, Axelsson, Hofmann, Kim and myself. For example, this subsumes past results on the Cauchy integral.

I bet this is not the last $T(b)$ theorem.