Geometry on spaces filled with projective lines

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Abstract. The complex plane $\mathbb{C}$ can be compactified by adding a point at infinity to give the Riemann sphere $S = \mathbb{C} \cup \{\infty\}$. Likewise, the n-dimensional complex Euclidean space can be compactified to form the complex projective space $\mathbb{C}P^n$, $\mathbb{C}P^1 = S$. $\mathbb{C}P^n$ can be identified with $\mathbb{C}^{n+1} - \{0\}/\sim$, where $u \sim v$ if and only if u and v are proportional. Coordinates $[z_0, z_1, \cdots, z_n]$ coming from $\mathbb{C}^{n+1}$ give homogeneous coordinates for $\mathbb{C}P^n$. For geometric problems $\mathbb{C}P^n$ is often preferred to $\mathbb{C}^n$ as an ambient space. For instance, an algebraic curve on $\mathbb{C}P^2$ is defined by a homogeneous polynomial in (z_0, z_1, z_2), and one can develop intersection theory to get Bezout's Theorem which determines the cardinality of the common zero set of two homogeneous polynomials on $\mathbb{C}P^2$, counting multiplicities, and the same cannot be done on $\mathbb{C}^2$.

On $\mathbb{C}P^n$ we have the notion of fractional linear transformations, which are precisely the bijective mappings on $\mathbb{C}P^n$ which are holomorphic in both directions. They transform projective lines to projective lines. There is a geometry on $\mathbb{C}P^n$, $n \geq 2$, in which the set of projective lines are treated as a distinguished set of holomorphic (i.e., complex-analytic) curves. From this perspective a basic question is to ask whether local information on these holomorphic curves determine global geometry. As a simple example of a positive answer to this question the following can be established: A holomorphic map defined on any connected open subset $D \subset \mathbb{C}P^n$ extends to a fractional linear transformation on $\mathbb{C}P^n$ whenever it maps open subsets of projective lines into projective lines. We will give an elementary explanation to this in the case of $\mathbb{C}P^2$. Other examples show even stronger positive answers to the question, e.g., when we consider the hyperquadric defined by $z_0^2 + \cdots + z_n^2 = 0$.